GRADED DEVIATIONS, RIGIDITY, AND THE KOSZUL PROPERTY

by

Michael DeBellevue

A DISSERTATION

Presented to the Faculty of

The Graduate College at the University of Nebraska

In Partial Fulfilment of Requirements

For the Degree of Doctor of Philosophy

Major: Mathematics

Under the Supervision of Professor Alexandra Seceleanu and Professor Mark Walker

Lincoln, Nebraska

August, 2022

GRADED DEVIATIONS, RIGIDITY, AND THE KOSZUL PROPERTY

Michael DeBellevue, Ph.D.

University of Nebraska, 2022

Adviser: Alexandra Seceleanu and Mark Walker

Differential graded algebras have played an important role in the study of infinite free resolutions over local commutative Noetherian rings. In particular, they have been a mechanism to import mathematical tools, such as the homotopy Lie algebra, from rational homotopy theory in order to study invariants of rings and growth of resolutions. More recently, differential graded algebras and the homotopy Lie algebra have also been employed to study infinite graded free resolutions over graded rings. In this setting the algebras are bigraded by homological degree and the internal degree coming from the ring of interest. Chapters 1, 2, and 3 of this work establish important theory of bigraded differential graded algebras in great generality. In chapter 4, we establish tools for computing the structure of these resolutions in particular internal degrees, yielding new results about the existence of certain long exact sequences. We then specialize to the study of $\mathbb{N}$-graded rings over fields. Motivated by a conjecture of Ferraro, in chapter 5 we prove results about the relationship between the structure of the homotopy Lie algebra and the Koszul and complete intersection properties. In chapter 6 We also prove variants of rigidity results which were previously known only in the local case.

DEDICATION

To my wife Aurora,

my parents, John and Lisa,

my advisors, Alexandra and Mark,

and to the many friends I have made along the way,

deepest thanks!

ACKNOWLEDGMENTS

I thank my wife Aurora DeBellevue for her constant love and support. Whenever I fall, Aurora is always there to pick me up.

I thank my parents, who poured so much of themselves into making my childhood the best that it could be. I thank my mother Lisa, who was my primary school teacher until I began college. I could not have succeeded academically without the strong educational foundation that we built together. I thank my father John, who taught me so much about the world, computers, self-reliance, and craft.

This work has grown under the careful supervision of my advisors Alexandra Seceleanu and Mark Walker. They have been invaluable in working through the many details of this work. They have been careful stewards of my career as a mathematical researcher by offering invaluable advice, supporting my travel and research activities, and teaching me so much mathematics that has yet to make its way into a class or textbook.

I thank my committee, and in particular my readers Brian Harbourne and Tom Marley, for their evaluation of my dissertation and defense, and their helpful feedback. I thank Greg Simon for being the outside member of my committee. He welcomed me into the musical community at UNL when I felt like an outsider. He provided invaluable career advice about music and academia, and invaluable instruction that developed my musical art. I also thank Tom Larson and Tyler White for their lessons and support in the music program. Without an artistic outlet, I could not have completed graduate school.

Marilyn Johnson's knowledge of various bureaucratic requirements is seemingly unending, which was an invaluable resource for preparing the various forms necessary for progressing through my program and advancing my research career. She went above and beyond by providing valuable reminders that kept me organized.

The algebra community at UNL and beyond has been tremendously welcoming, and this community is the principle reason that I continued in my study of mathematics. In particular, I thank David Jorgensen, who sparked my love of commutative algebra and whose help was instrumental to succesfully beginning my graduate career. I thank Roger and Sylvia Wiegand for always working to create a friendly community and for their wonderfully positive outlook. My interest in commutative algebra was nurtured by the many conversations that I had with graduate students in this area. I am especially thankful to Nick Packauskas and Josh Pollitz, the shepards of this community when I began graduate studies, for welcoming me as a friend and research colleague. I am also thankful to Matthew Bachmann, Julie Geraci, Mohsen Gheibi, Amadeus Martin, David Liebermann, and Dakota White for their work building the community at UNL and for helpful conversations about algebra.

Special thanks to Ben Briggs, and additional special thanks to Josh Pollitz, who readily shared their expertise in areas relating to my research.

I also thank my many friends who supported me throughout graduate school. There are far too many to name them all, but in particular, I thank Ana Wright, Kaitlyn Tademy, Audrey Goodnight, and Matt Reichenbach.

Table of Contents

Chapter 1

Introduction

The philosophy which began homological algebra is to use sequences of matrices between free modules (often vector spaces), which organize together into a structure called a *chain complex*, to create numerical measurements of mathematical objects. In good cases, these measurements allow for mathematical objects to be compared with great precision using computational tools from linear algebra. The discipline began in topology, in which the number of "n-dimensional holes" in a topological space is measured by the vector space dimension of the n'th piece of a chain complex associated to the space. Homological algebra was then imported into many other areas of mathematics, including commutative algebra [38]. When R is a local Noetherian ring, a chain complex of particular interest is the minimal resolution of the residue field k of R by free R-modules. The sizes of the free modules (their R-rank) in this chain complex is a measurement of the complexity of the relations among the generators of the ring's maximal ideal.

Differential graded algebras further enrich the structure of a chain complex by imposing a multiplication rule. Again, they were first used in topology, and were later imported to the study of local rings by John Tate [37]. Specifically, the resolution of the residue field is inductively constructed by adjoining algebra generators in increasing homological degrees. The ranks of the free modules in the resulting resolution is determined by the number of algebra generators by a combinatorial formula. Hence, the introduction of an algebra structure to the resolution of k over R allows for the growth of resolutions to be studied by counting the number of algebra generators in each homological degree. These numbers form a family of numerical invariants $\{\varepsilon_i(R)\}_{i\geq 1}$ called the *deviations of* R.

A particularly spectacular example of how deviations can measure resolutions is when R is a complete intersection (a quotient of a regular ring by a regular sequence). In this situation, the (typically infinite) data of the minimal resolution of the residue field is entirely encoded in the finite data of the first three deviations $\varepsilon_0(R), \varepsilon_1(R), \varepsilon_2(R)$. Soon after Tate introduced differential graded algebra resolutions, his student Assmus proved as part of his dissertation work that the converse is true also [2]. Namely, when the the deviations vanish in homological degrees greater than three, R must be a complete intersection.

When R fails to be a complete intersection, the severity of this failure can be measured by the deviations $\varepsilon_i(R)$ for $i \geq 3$. If many are non-zero and large in magnitude, this indicates that R is far from being a complete intersection. In a line of work begun by Gulliksen [23], deviations have been shown to be *rigid*, in the sense that either R is a complete intersection and all higher deviations vanish, or every single deviation must be non-zero [25], and in fact the sequence of deviations must grow exponentially [4]. This shows that the homological behavior of complete intersections is radically different from that of other rings.

Differential graded algebras can be used to gain even greater insight into the structure of the homological algebra over the ring R. There exists an $\mathbb{N}$-graded Lie algebra called the *homotopy Lie algebra of* R, denoted $\pi^*(R)$, which completely determines the structure of the Ext algebra of R. Specifically, composition of morphisms gives a multiplicative structure on the Ext algebra

$$\mathrm{Ext}_R^*(k,k) = \oplus_{i\in\mathbb{N}} \mathrm{Ext}_R^i(k,k).$$

This multiplication is typically not commutative, so the (graded) commutator defines a non-trivial bracket turning $\mathrm{Ext}_R^*(k,k)$ into a Lie algebra. $\pi^*(R)$ is a Lie subalgebra with respect to this bracket, and $\mathrm{Ext}_R^*(k,k)$ is the universal envelope of $\pi^*(R)$. This means that $\mathrm{Ext}_R^*(k,k)$ is constructed from $\pi^*(R)$ by imposing that its commutator agrees with the bracket of $\pi^*(R)$, and no other conditions. The deviations enumerate the dimensions graded pieces of the homotopy Lie algebra: $\varepsilon_i(R) = \dim_k \pi^i(R)$ for each i [36].

Again, there is inspiration from topology: the homotopy Lie algebra is the analogue of the rational homotopy groups $\pi^1(X), \pi^2(X), \ldots$ of a topological space X, justifying the name [6]. There are strong restrictions on its structure (for example see [20, Theorem A]) which have made it a vital tool for proving variants of rigidity theorems such as [6, Theorem D]. Recently, the homotopy Lie algebra was employed by Briggs to close a long-standing conjecture of Vasconcelos on the conormal module [16, Theorem A].

More recent work has expanded the use of differential graded algebras to the study of graded rings. Typical cases are quotients of the polynomial ring $k[x_1,\ldots,x_n]/I$ over a field k, with I generated by polynomials which are homogeneous with respect to a grading determined by an assignment of the degrees of each variable x_i. Two particularly studied cases are

- $\deg(x_i) = \vec{e}_i$ is the standard basis vector of $\mathbb{N}^n$, in which case homogeneous polynomials are necessarily monomials and the $\mathbb{N}^n$-graded ring R is said to be a *monomial algebra*

- $\deg(x_i) = 1$, in which case R is a *standard* $\mathbb{N}$*-graded ring*.

Avramov [7] showed that certain deviations of a monomial ring R and a Lie subalgebra of $\pi^*(R)$ may be calculated using some combinatorial data of a certain poset associated to the defining ideal I. This was expanded by Berglund [13].

Avramov and Peeva [10] showed that vanishing of certain deviations of an $\mathbb{N}$-graded ring are associated to the Koszul property, which is a duality property enjoyed by some $\mathbb{N}$-graded rings which is especially important in mathematical physics[29, Chap. 13, 7]. Ferraro was interested in deviations of $\mathbb{N}$-graded rings due to a connection to the existence of test modules for the Koszul property [21, Question 2.4], and made a few conjectures about them (Question 3.3, Question 3.4, loc. cit.) motivated by the results of Avramov and Peeva and the rigidity theory of Gulliksen.

A systematic treatment of differential graded algebras, deviations, and homotopy Lie algebras for more general gradings is absent in the literature. To account for the internal grading, we name these objects *differential bigraded algebras* and *bigraded homotopy Lie algebras*. To them are associated a set of *graded deviations* $\varepsilon_{i,j}(R)$, where i ranges over all homological degrees, and j ranges over all internal degrees of the gradation of R. This work begins by showing that the aforementioned results proven for local rings hold also for many graded rings. We then discuss a number of completely new results involving graded deviations, which we will highlight in the following overview of the remaining chapters.

Chapter 2 defines the category of differential bigraded algebras and establishes some technical results that will be needed later. Chapter 3 features theorem 3.1.5 and theorem 3.1.7, which establish the existence of *acyclic closures* and *minimal models*, two algebra resolutions which are used to define the deviations, for arbitrary graded rings. Some minimality conditions are defined, which allow for these algebra resolutions to be detected. We also describe a procedure for minimizing differential bigraded algebras, which is an essential computational tool employed heavily in chapter 5. In chapter 4, we define deviations, and construct the homotopy Lie algebra in a more restrictive case. We show that it is unique in theorem 4.3.6:

Theorem. *Suppose that $\mathbb{D}$ is a commutative cancelative monoid that has no non-trivial units, and let R be a $\mathbb{D}$-local ring with $R_0 = k$ a field. Then $\pi^*(R)$ is unique up to isomorphism of bigraded Lie algebras.*

We also discuss some how some known results on long exact sequences of homotopy Lie algebras extend to the graded setting.

The latter part of this work is entirely new. It features a method of calculating deviations, connections between vanishing of deviations and structural properties of rings, and a rigidity theory for graded deviations which has consequences for other invariants of graded rings.

chapter 5 features theorem 5.1.7, which provides a method for calculating deviations using any differential bigraded algebra which satisfies certain technical conditions. We state a slightly simplified version below:

Theorem. *Let $\mathbb{D}$ be a commutative cancelative monoid and R be a $\mathbb{D}$-local ring and $R_0 = k$ a field. Let $k[Y]$ be a $\mathbb{D}$-local semifree extension with $H_0(Q[Y]) = R$. Let $D \subset \mathbb{D}$ be summand closed, and suppose $H_{\geq 1}(Q[Y])_D = 0$ and that $k[Y]$ is absolutely minimal in all degrees coming from D. Then there is a minimal model $k[X]$ of R such that $\#X_{i,d} = \#Y_{i,d} = \varepsilon_{i,d}(R)$ for all $i \in \mathbb{N}$ and $d \in D$.*

Several corollaries follow for relating deviations of rings R and S connected by a graded morphism $\varphi\colon R \to S$.

Chapter 6 studies the connection between vanishing of certain deviations and the Koszul and complete intersection properties. Theorem 6.1.1 uses deviations to study the transfer of the Koszul property along a graded homomorphism. Theorem 6.2.4 provides a connection between the vanishing of "off-diagonal" deviations and a hybrid of the Koszul and complete intersection properties. A direct consequence is corollary 6.2.6, which generalizes [10, Theorem 2] and answers [21, Question 3.3]:

Theorem. *Let S be an $\mathbb{N}$-local algebra with $S_0 = k$, and suppose that $\varepsilon_{ij}(S) = 0$ when $j \neq i$ and $j \geq 3$. Then $S \cong (Q \otimes_k P)/(f_1, \ldots, f_c)$ with Q standard-graded Koszul, P a polynomial ring, and $f_1, \ldots, f_c$ a regular sequence. If S is standard graded then $S \cong Q/(f_1, \ldots, f_c)$ with Q Koszul and $f_1, \ldots, f_n$ a regular sequence.*

In chapter 7 we study whether the off-diagonal deviations appearing in the above theorem exhibit rigid behavior. As with classical rigidity results, the aim is to measure the gap in homological behavior between rings satisfying the conclusion of the prior theorem, and rings which do not. Theorem 7.2.3 establishes a rigidity theorem for $\mathbb{N}$-graded deviations. Corollary 7.2.4 is analogous to the classical theorem of Gulliksen [23], but only for odd homological degrees:

Theorem. *Let R be an $\mathbb{N}$-local algebra with $R_0 = k$. If $\varepsilon_{ij}(R) = 0$ for $i \neq j$ and $i \gg 0$, then $\varepsilon_{ij}(R) = 0$ for $i \neq j$ and all odd $i \geq 3$.*

We then apply this result to study the asymptotic properties of a numerical invariant of $\mathbb{N}$-graded rings called *slope*. Theorem 7.3.6 makes progress towards answering a question of Conca [30, p. 9.4]:

Theorem. *Let R be a graded k-algebra. Then either* $\operatorname{limslope}_R(k) = \operatorname{slope}_R(k)$ *or the following hold:*

1. *There exists an odd integer l so that* $\operatorname{slope}_R(k) = (t_l(k) - l)/l$. *In other words, the slope is attained in some degree l.*

2. *For any l satisfying (1), we have* $\operatorname{slope}_R(k) > \operatorname{limslope}_R(k) \geq (t_l(k) - l)/(l+1)$.

The above constructions have been generalized to the *relative case* of a morphism $\varphi\colon R \to S$ of local rings which maps the maximal ideal of R into the maximal ideal of S. Specifically, the deviations $\varepsilon_i(\varphi)$ and the homotopy Lie algebra $\pi^*(F^\varphi)$ of φ may be defined using a slightly technical construction which we outline in the graded setting in chapter 4. The classical *absolute case* of the deviations and homotopy Lie algebra of a ring is recovered by considering a presentation $Q \to \widehat{R}$ of the completion of R as a quotient of a regular ring. In this work, we focus on establishing all the aforementioned results in the relative case due to its greater generality. Our summary above featured the absolute case to avoid excessively technical discussion.

Chapter 2

Differential-Bigraded Algebra

Over a regular ring, the celebrated Auslander-Buchsbaum-Serre theorem states that all free resolutions are finite. Hence, provided an algorithm exists to calculate the generators of kernels, free resolutions of any module can be precisely calculated in finite time. The desire to understand the structure of infinite resolutions over other local rings has led to the development of a variety of tools under the umbrella of differential graded algebra, namely differential-graded algebra resolutions, acyclic closures, and differential-graded Lie algebras (in particular, the homotopy Lie algebra attached to a ring or a ring homomorphism). These tools have been used to investigate a number of prominent homological conjectures, chief among them Quillen's conjecture on the vanishing of cotangent cohomology [5] and, more recently, Vascencelos' conjecture on the conormal model [17]. Suitable complete references for differential graded algebra may be found in the books of Gulliksen and Levin [22] and Avramov [8].

These tools were originally developed for the study of local algebra (i.e., local rings and local homomorphisms between them). An essential component for the development of the theory is Nakayama's lemma, which holds for the category of finitely generated modules over a local ring. As a version of Nakayama's lemma holds for the category of graded modules over a graded ring, many homological constructions proceed identically. Consequently, DG algebra tools have been employed for graded rings in various contexts (see examples [10], [13]) However, a complete independent development in a general graded setting is unknown to the present author, so we proceed with a self-contained treatment.

2.1 Monoid Graded Rings

We begin by establishing some notation about graded rings. To state results in their natural level of generality, we will allow the grading to be provided by an arbitrary commutative cancelative monoid $\mathbb{D}$, written additively. In typical examples, $\mathbb{D}$ is $\mathbb{N}$ or $\mathbb{N}^l$. We will typically need to enlarge allowable gradings by passing to the group completion.

Definition 2.1.1. Let $\mathbb{D}$ be a commutative cancelative monoid. The *group completion of* $\mathbb{D}$, denoted $G(\mathbb{D})$ or just G when $\mathbb{D}$ is understood, is the group obtained by adjoining a formal inverse for every non-unit of $\mathbb{D}$.

In this section, R will be a commutative Noetherian ring, and will be $\mathbb{D}$-graded, the definition of which we recall below.

Definition 2.1.2. Let $\mathbb{D}$ be a monoid.

1. A $\mathbb{D}$-*graded ring* R is a ring whose underlying abelian group is equipped with a direct sum decomposition $R = \bigoplus_{d\in\mathbb{D}} R_d$ satisfying $R_d R_e \subset R_{d+e}$ for all $d, e \in \mathbb{D}$.

2. A *morphism of* $\mathbb{D}$-*graded rings* is a ring homomorphism $\varphi\colon R \to S$ satisfying $\varphi(R_d) \subset S_d$ for all $d \in \mathbb{D}$ (in other words, morphisms always preserve $\mathbb{D}$ degrees).

3. A G-*graded module* M is an R-module equipped with a direct sum decomposition $M = \bigoplus_{d\in G} R_d$ satisfying $R_d M_e \subset M_{d+e}$ for all $d, e \in G$.

4. A $\mathbb{D}$-*graded module* is a G-graded module M further satisfying that $M_d = 0$ whenever $d \notin \mathbb{D}$

5. A *map of G-graded modules* $f\colon M \to N$ is an R-linear map satisfying $f(M_d) \subset N_d$ for all $d \in D$ (in other words, morphisms always preserve G degrees).

6. For $d \in G$ and a G-graded R module M, the *shift of M by d*, denoted $M(d)$, is the module $M(d) = \bigoplus_{e\in G} M_{e+d}$.

7. Associated to the shift is an R-linear function $(d) : M \to M(d)$ which takes $m \in M_e$ and reinterprets it as an element of M_{e+d}, denoted $m(d)$.

The structures of graded R modules and graded rings combine to form algebras.

Definition 2.1.3. A $\mathbb{D}$-graded R-algebra is a $\mathbb{D}$-graded ring S equipped with a $\mathbb{D}$-graded ring homomorphism from R to the center of S.

Definition 2.1.4. Let $\mathbb{D}$ be a (commutative, cancelative) monoid. A subset $S \subset \mathbb{D}$ is an *ideal* if it is closed under the addition of $\mathbb{D}$. S is *summand-closed* or a *summand set* if $s + s' \in S$ implies that $s \in S$ and $s' \in S$.

While we will not make use of the following fact, we include it to provide further context for the above definition.

Proposition 2.1.5. *$S \subset \mathbb{D}$ is summand-closed if and only if $\mathbb{D} \setminus S$ is an ideal.*

The following example is commonly encountered:

Example 2.1.6. $\{0, 1\}^l \subset \mathbb{N}^l$, the *set of squarefree multidegrees*, is summand-closed.

Resolutions are commonly studied in the context of local rings, which allows for the use of Nakayama's lemma. In the graded case, there is an analogue of locality.

Definition 2.1.7. R is a $\mathbb{D}$-*graded local ring* or $\mathbb{D}$-*local* if the ideal generated by all homogeneous non-units of R is proper. In this case, the ideal generated by all homogeneous non-units is called the *irrelevant ideal of* R and is denoted $\mathfrak{m}_R$. A morphism $\varphi\colon R \to S$ of $\mathbb{D}$-local rings satisfies $\varphi(\mathfrak{m}_R) \subseteq \mathfrak{m}_S$. R is a $\mathbb{D}$-*graded field* if every homogeneous element is invertible (i.e., $\mathfrak{m}_R = 0$ and R_0 is a field).

Even in this level of generality, Nakayama's lemma applies which allows for the construction of minimal resolutions. Its proof may be found in [28, Theorem 3.4] and [26, Proposition 2.30] (the latter for the case of Abelian groups, but since $\mathbb{D}$ is cancelative the result may be applied to $G(\mathbb{D})$-graded modules).

Lemma 2.1.8. *Let M be a finitely generated $G(\mathbb{D})$-graded module and R be a $\mathbb{D}$-local ring with irrelevant ideal $\mathfrak{m}$. Then,*

1. *$\mathfrak{m}M = M$ implies $M = 0$,*
2. *$m_1, \ldots, m_n$ generate M if and only if their images in $M/\mathfrak{m}M$ generate $M/\mathfrak{m}M$,*
3. *Any minimal generating set of M has the same length depending only on M.*

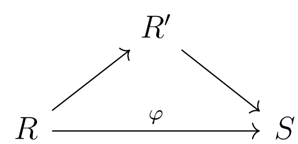
This ensures that the following is well-defined.

Definition 2.1.9. For a finitely generated $G(\mathbb{D})$-graded R module M, the minimal number of homogeneous generators is denoted by $\mu(M)$.

2.1.1 Presentability

For the classically-studied case of local rings (in the ordinary sense, that there is a unique maximal ideal), morphisms $\varphi\colon R \to S$ must map the maximal ideal $\mathfrak{m}_R$ of R to the maximal ideal $\mathfrak{m}_S$ of S. The famous Cohen structure theorem states that if R is complete, it admits a presentation $R \cong Q/I$ with $I \subset \mathfrak{m}_Q^2$ and Q regular, and the map $Q \to R$ is one of local rings. In the equicharacteristic case, Q is a power series ring in some number of variables.

In the relative situation, a local homomorphism $\varphi\colon R \to S$ with S complete admits a *Cohen factorization*, which is a commutative diagram of local rings



with R' a complete ring, R' flat over R, and $R'/(\mathfrak{m}_R R')$ a regular ring. Such a factorization is called *minimal* if $\operatorname{edim} R'/(\mathfrak{m}_R R') = \operatorname{edim} S/(\mathfrak{m}_S S)$. Minimal Cohen factorizations always exist when S is complete by [9, (1.1)]

In the graded setting, a natural analogue of the power series rings Q and $R'/(\mathfrak{m}_R R')$ is the polynomial ring in some number of variables. Unfortunately, this notion proves to be more restrictive than in the local case. In other words, not all rings and homomorphisms admit presentations in the manner described above, as demonstrated by the following example.

Example 2.1.10. Let $R = k[t, t^{-1}]$ be $\mathbb{Z}$-graded with $\deg(t) = 1$. Let $R \cong Q/I$ be any presentation of R as a quotient of a polynomial ring over a field, and let $x, y \in Q$ be pre-images of t, t^{-1}. Since the isomorphism is graded, Q must be $\mathbb{Z}$-graded so that $\deg(x) = 1$ and $\deg(y) = -1$. Both xy and $xy - 1$ are homogeneous non-units with respect to this grading, so the ideal generated by all homogeneous non-units of Q fails to be proper. Hence Q is not $\mathbb{Z}$-local. Therefore, R can not be described as a quotient of a $\mathbb{D}$-local polynomial ring.

Consider the natural inclusion $\varphi\colon k \hookrightarrow R$. Since $\mathfrak{m}_k = 0$, the map $k \to R'$ in any factorization of φ as described above satisfies $R'/(\mathfrak{m}_k R') = R'$. By the same argument as in the preceding paragraph, R' can not be a polynomial ring and be $\mathbb{D}$-local. Hence

φ can not be factored in a manner analogous to Cohen factorizations.

With this in mind, we introduce the following notions:

Definition 2.1.11. A $\mathbb{D}$-local ring R is *presentable* if there exists a $\mathbb{D}$-local polynomial ring Q over a complete regular local ring Q_0 so that $R \cong Q/I$ with $I \subset \mathfrak{m}_Q^2$. A morphism $\varphi\colon R \to S$ is *factorizable* if there exists a commutative diagram of $\mathbb{D}$-local rings

$$\begin{array}{ccccc} & & R' & & \\ & \nearrow & & \searrow & \\ R & & \xrightarrow{\quad\varphi\quad} & & S \end{array}$$

with R' a $\mathbb{D}$-local ring, R' flat over R, and $R'/(\mathfrak{m}_R R')$ a polynomial ring over a regular ring $(R'/(\mathfrak{m}_R R'))_0$. A diagram above is called a *factorization* of φ.

Note that while it is possible that R' in the above definition is nothing other than $R[x_1, \ldots, x_n]$, it need not be. Of course, if $\varphi_0\colon R_0 \to S_0$ is a local morphism admitting a Cohen factorization $R_0 \to R_0' \to S$ in which R_0' is not a polynomial ring over R_0, then R' will fail to decompose in such a manner. However, even when R and S are algebras over fields, such a decomposition of R' may fail to hold; An example of a factorizable map in which R' is not a polynonomial ring over R is a field extension $k \to l$. However, when R and S are algebras over fields, this is the only way the decomposition can fail, meaning that we can always take R' to be a polynomial ring over $R \otimes_k l$. This will be shown in proposition 2.1.17, but we first establish a few more necessary definitions and lemmas.

Definition 2.1.12. Let R and S be $\mathbb{D}$-local rings with $R_0 = k$ and $S_0 = l$. Let $R \to R' \to S$ be a factorization of a $\mathbb{D}$-local map $R \to S$, and suppose $R'/(\mathfrak{m}_R R') \cong l[x_1, \ldots, x_n]$. The factorization is *standard* if $R' \cong (R \otimes_k l)[x_1, \ldots, x_n]$.

We can also require that presentations and factorizations are as efficient as possible, in the sense that they use the fewest variables.

Definition 2.1.13. Let R be a presentable $\mathbb{D}$-local ring. Then the *embedding dimension* of R, denoted $\operatorname{edim}(R)$, is the minimal number of generators, $\mu(\mathfrak{m}_R)$, of the maximal ideal. A presentation $Q \to R$ is *minimal* if $\dim(Q) = \operatorname{edim}(R)$. A factorization $R \to R' \to S$ is *minimal* if $\dim R'/(\mathfrak{m}_R R') = \operatorname{edim} S/(\mathfrak{m}_R S)$.

Factorizations allow for the definitions of various singularity types of rings to be generalized to singularity types of ring homomorphisms. We define one in particular which is of particular importance to this work.

Definition 2.1.14. Let $\varphi\colon R \to S$ be a $\mathbb{D}$-local homomorphism which admits a minimal standard factorization $R \to R' \xrightarrow{\widetilde{\varphi}} S$. φ is a *complete intersection homomorphism* if a $\ker(\widetilde{\varphi})$ is minimally generated by a regular sequence.

Example 2.1.15. Let k be a field and $R = k[x]/(x^2)$ and $S = k[x,y]/(x^2,y^2)$ be standard $\mathbb{N}$-graded (i.e. , $\deg(x) = \deg(y) = 1$). The natural inclusion of R into S factors as

$$k[x]/(x^2) \to k[x,y]/(x^2) \to k[x,y]/(x^2,y^2)$$

where the latter map is the natural surjection. y^2 is regular in $k[x,y]/(x^2)$, so the natural inclusion $R \hookrightarrow S$ is a complete intersection homomorphism.

To avoid issues like those arising in example 2.1.10, we will typically work under some additional assumptions on the structure of $\mathbb{D}$, R, and $R \to S$ to ensure presentability and factorizability.

Proposition 2.1.16. *Suppose $\mathbb{D}$ has no non-trivial units. Then,*

1. *Any $\mathbb{D}$-field is a field (in the ordinary sense).*

2. Let $d_1, \ldots, d_n \in \mathbb{D}$ be non-zero and let

$$Q = k[x_1, \ldots, x_n, \mid \deg(x_i) = d_i]$$

be the polynomial ring over a field k. Then Q is $\mathbb{D}$-local.

Proof. (1) holds by contrapositive: if k is a $\mathbb{D}$-field and $\lambda \in k$, then $\deg(\lambda^{-1}) = -\deg(\lambda)$, forcing $\deg(\lambda)$ to be a unit and hence $\deg(\lambda) = 0$ by assumption.

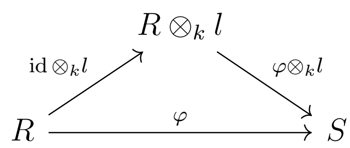
For (2), each $x_i \in Q$ is homogeneous and is a non-unit, and so $(x_1, \ldots, x_n)$ is contained in the ideal of all homogeneous non-units of Q. On the other hand, since $\mathbb{D}$ has no non-trivial units, we have $\deg(m) \neq 0$ for any non-scalar monomial. This implies that $Q_0 = k$, and hence Q is $\mathbb{D}$-local with $\mathfrak{m}_Q = (x_1, \ldots, x_n)$ □

Proposition 2.1.17. *Suppose $\mathbb{D}$ has no non-trivial units, and let R be a $\mathbb{D}$-local ring with $R_0 = k$ a field. Then R is minimally presentable. If $\varphi \colon R \to S$ is a $\mathbb{D}$-local homomorphism with S a commutative Noetherian ring with $S_0 = l$ a field, then φ is minimally factorizable, and we can require that the factorization is standard. Furthermore, a minimal factorization $k \to Q \to R$ of the inclusion of k into R yields a minimal presentation of R.*

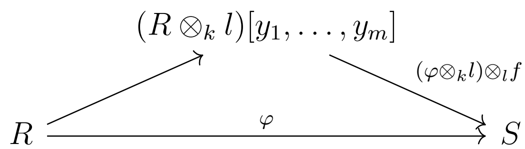
Proof. Since R is Noetherian (a global assumption for this section), $\mathfrak{m}_R$ is finitely generated. Let $\bar{x}_1, \ldots, \bar{x}_n$ be a minimal generating set. Set $Q = k[x_1, \ldots, x_n]$ with $\deg(x_i) = \deg(\bar{x}_i)$. The map determined by sending x_i to $\bar{x}_i$ determines a presentation $R \cong Q/I$, and the map $Q \twoheadrightarrow R$ is a $\mathbb{D}$-local homomorphism, and $\dim Q = \operatorname{edim} R$. Hence R is minimally presentable.

Since S is Noetherian, $S/(\mathfrak{m}_R S)$ is also, and hence $S/(\mathfrak{m}_R S)$ is presentable by the above argument. Suppose $l[y_1, \ldots y_m] \to S/(\mathfrak{m}_R S)$ is a presentation of $S/(\mathfrak{m}_R S)$, and let $f\colon l[y_1, \dots y_m] \to S$ be a ring homomorphism determined by lifting of the images of $y_1, \dots, y_m$ to S.

Extending scalars yields a commutative diagram



Tensoring over l yields the following diagram



The ring $R' = (R \otimes_k l)[y_1, \dots, y_m]$ and the diagram above satisfy all the required properties of a minimal standard factorization of φ:

1. R' is a tensor product of $\mathbb{D}$-local rings over a field, and so is $\mathbb{D}$-local

2. $R' \to S$ is surjective

3. Since $R \mapsto R'$ is a composition of flat maps, it is flat

4. $R'/(\mathfrak{m}_R R') \cong l[y_1, \dots, y_m]$ is a polynomial ring

5. $m = \operatorname{edim} S/(\mathfrak{m}_R S)$, so the factorization is minimal

For the final remark, note that in a minimal factorization $k \to Q \to R$, Q is a polynomial ring with $\operatorname{edim}(Q) = \operatorname{edim}(R)$ and hence $Q \to R$ is a minimal presentation.

$\square$

Proposition 2.1.18. *Suppose $\mathbb{D}$ has no non-trivial units and that $R_0 = k$ and $S_0 = l$ are fields. Let $\varphi\colon R \to S$ be a $\mathbb{D}$-local homomorphism. A minimal standard factorization of φ is unique up to isomorphism of factorizations, in the following sense: if $R \to T \xrightarrow{f} S$ and $R \to U \xrightarrow{g} S$ are two minimal factorizations of φ, then there exists a commutative diagram*

$$\begin{array}{ccc} R & \xrightarrow{=} & R \\ \downarrow & & \downarrow \\ T & \xrightarrow{\cong} & U \\ \downarrow & & \downarrow \\ S & \xrightarrow{=} & S \end{array}$$

Proof. Set $S' = S/(\mathfrak{m}_R S)$, and let $n = \operatorname{edim}(S/(\mathfrak{m}_R S))$. Suppose $T/(\mathfrak{m}_R T) = l[x_1, \dots, x_n]$ and $U/(\mathfrak{m}_R U) = l[y_1, \dots, y_n]$. Then $f(x_1), \dots, f(x_n)$ and $g(y_1), \dots, g(y_n)$ are minimal generating sets for $\mathfrak{m}_{S'}$, and so their images form vector space bases of $\mathfrak{m}_{S'}/\mathfrak{m}_{S'}^2$. Hence we obtain expressions $f(x_i) = \sum_j (a_{ij} + g(b_{ij}))g(y_j)$ for $i = 1, \dots, n$ in which $A = (a_{ij}) \in \operatorname{Mat}_{n \times n}(k)$ is an invertible matrix and $g(b_{ij}) \in \mathfrak{m}_R$. Let $\psi\colon k[x_1, \dots, x_n] \to k[y_1, \dots, y_n]$ be the k-algebra homomorphism defined by sending x_i to $\sum_j (a_{ij} + b_{ij})y_j$.

The composition of ψ with the projection to $(y_1, \dots, y_n)/(y_1, \dots, y_n)^2$ is the k-span of $\{\sum_j a_{ij} y_j\}_{i=1,\dots,n}$ which is a basis since (a_{ij}) is an invertible matrix, and hence ψ is surjective and therefore an isomorphism.

Since the presentations are standard, we have that

$$T \cong (R \otimes_k l)[x_1, \dots, x_n] \cong (R \otimes_k l) \otimes_l l[x_1, \dots, x_n].$$

and similarly $U \cong (R \otimes_k l) \otimes_l l[y_1, \dots, y_n]$. The map $(R \otimes_k l) \otimes_l \psi\colon T \to U$ is the desired isomorphism of presentations. □

Remark 2.1.19. Any minimal presentation of R is a standard minimal factorization of the map $k \to R$, so the above shows that presentations are minimal up to isomorphism.

2.2 Bigraded Rings

Remark 2.2.1. In the classical study of DG algebras, R is an $\mathbb{N}$ graded ring concentrated in degree zero, and a multiplicative structure is defined upon a free resolution F of a quotient ring of R by choosing an R-linear map $F \otimes_R F \to F$. As F is a sequence of R modules, rather than the direct sum of all of them, this clashes with the traditional notion of an algebra introduced in definition 2.1.3. Some authors in homological algebra (cf. [8, Ch. 1]) refer to these objects as graded algebras regardless. We will adopt the convention that "graded algebra" always mean a direct sum in the ordinary sense. This is a purely notational decision, for the assignments $\oplus_n A_n \mapsto \{A_n\}_n$ and $\{A_n\}_n \mapsto \oplus_n A_n$ are inverse functors exhibiting an isomorphism of categories. As collateral damage, "chain complex" shall also mean a direct sum of modules (see definition 2.3.1)

Definition 2.2.2.

1. A *homologically* $\mathbb{D}$*-bigraded ring* ("bigraded ring" for short) A is an $\mathbb{N} \times \mathbb{D}$-graded ring.

2. A *homologically* $G(\mathbb{D})$*-bigraded left* A*-module* ("bigraded left A module" for short) is an $\mathbb{Z} \times G(\mathbb{D})$-graded left module.

3. When $N \subset \mathbb{Z}$, $D \subset G(\mathbb{D})$, $i \in \mathbb{Z}$ and $d \in G(\mathbb{D})$, (N, d) denotes the set $\{(n, d) \,|\, n \in \mathbb{N}, \}$ and (i, D) denotes the set $\{(i, e) \,|\, e \in D\}$.

4. The use of the character "*" will always represent the total set of indices understood by the context. In particular, for $i \in \mathbb{Z}$ and $d \in G(\mathbb{D})$, $(i, *)$ denotes the set $\{(i, d) \,|\, d \in \mathbb{D}\}$ and $(*, d)$ denotes the set $\{(i, d) \,|\, i \in \mathbb{N}\}$.

5. For a subset $S \subset \mathbb{N} \times \mathbb{D}$, A_S denotes the set $\oplus_{(i,d) \in S} A_{(i,d)}$. Similarly, when $S \subset \mathbb{Z} \times G(\mathbb{D})$, and M is a bigraded left A module, $M_S = \bigoplus_{(i,d) \in S} M_{(i,d)}$

6. For a bigraded left A-module M, an element $m \in M_{i,*}$ is *homogeneous of homological degree* i, written $|m| = i$, and an element $m \in M_{*,d}$ is *homogeneous of internal degree* d, written $\deg(m) = d$. An element $m \in M_{i,d}$ is *bihomogeneous of homological degree* i *and internal degree* d.

7. For brevity, $M_{i,*}$ and $M_{N,*}$ will be frequently abbreviated M_i and M_N.

Remark 2.2.3. The use of the phrase "homological degree" is in reference to our ultimate aim of studying graded free resolutions, which are complexes of $\mathbb{D}$-graded modules indexed by $\mathbb{N}$.

Remark 2.2.4. The most typical way that the notation introduced in definition 2.2.2 is used is to discuss truncations of bigraded modules to particular homological degrees, such as $M_{\geq i}$ and $M_{\leq i}$, or to particular internal degrees, such as $M_{*,\geq d}$ (when $\mathbb{D}$ is equipped with some partial order) or $M_{*,D}$ (more generally).

The categories of complexes and of graded modules both come equipped with shifting (also called suspension) functors, each of which have their own traditional notation. Furthermore, they each have their own conventions regarding how maps between graded objects are specified. In this work, an attempt will be made to respect both conventions, despite some resulting notational peculiarities.

Definition 2.2.5. Let M be a bigraded left A-module, and let $(i,d) \in \mathbb{Z} \times G(\mathbb{D})$. The *shift of* M *up by* (i,d) is the module $\Sigma^i M(-d)$ with underlying graded pieces $\Sigma^i M(-d)_{j,e} = M_{j-i,e-d}$. Associated to the shift is the set map which takes an element $m \in M_{j,e}$ and reinterprets it as an element $\Sigma^i m(d)$ of $M_{j-i,e-d}$. The left module action of A upon $\Sigma^i M(-d)$ is the same as the left module action of A upon M, but twisted by the homological degree:

$$a(\Sigma^i m(d)) = (-1)^{i|a|}\Sigma^i(am)(d).$$

Definition 2.2.6. When M and N are bigraded modules over a bigraded ring A and $i \in \mathbb{Z}$, an morphism $f\colon M \to N$ of *homological degree* i is a morphism of underlying $\mathbb{D}$-graded A-modules which sends $M_{j,d}$ to $M_{j+i,d}$. It is A-linear if it satisfies the A-linearity condition $f(am) = (-1)^{d|a|}af(m)$.

Remark 2.2.7. The various sign conventions above arise from placing a DG-category structure on the category of $\mathbb{D}$-graded modules. Since the data of the $\mathbb{D}$-grading is internal to the starting category, the resulting signs do not depend on the $\mathbb{D}$-grading[1]. See [27] for a description of a DG-category structure.

In order to define algebras over a bigraded ring we need a notion of commutativity, so that the tensor product remains in the appropriate module category. As with ordinary rings, the center may be defined, but in order to avoid a clash with the notation $Z(A)$ for the cycles of A, we adopt the unconventional notation center(A).

Definition 2.2.8. For a bigraded ring A, center(A) is the bigraded subring generated by all bihomogeneous elements of A satisfying $ab = (-1)^{|a||b|}ba$ and $a^2 = 0$ whenever $|a|$ is odd. A is *strictly graded-commutative* if center$(A) = A$. In this case, the *canonical bimodule structure* of a left A-module is the assignment $ma = (-1)^{|m||a|}am$. When A is strictly graded-commutative, a bigraded ring B is a *homologically* $\mathbb{D}$-*bigraded* A-*algebra* if there is a bigraded ring homomorphism $A \to$ center(B) turning B into an A-bimodule.

[1]This also further explains the use of the phrase “internal degree”

To avoid the overly-long assumption statements in proceeding material, rings and algebras will be assumed to be strictly graded-commutative and all bigraded left modules will be given the canonical bimodule structure unless explicitly stated otherwise. We will refer to bigraded modules with this canonical bimodule structure simply as modules.

2.3 Bigraded Differential Algebra

Definition 2.3.1. When R is a $\mathbb{D}$-graded ring, it may be viewed as a bigraded ring concentrated in homological degree zero, which provides a sensible notion of bigraded R modules. A chain complex over R is a bigraded module M equipped with a square-zero map $\partial_M \colon M \to M$ of homological degree -1 (preserving internal degrees).

The differentials of chain complexes M and N may be combined in order to create a differential on the tensor of the complexes. Ensuring that the differential squares to zero requires the introduction of a sign, a phenomenon common throughout homological algebra. Chain maps from the tensor product complex define multiplicative structures on resolutions which are compatible with the differential, resulting in the substitution of differential-bigraded algebras and modules for rings and chain complexes.

Definition 2.3.2. Let R be a $\mathbb{D}$-graded ring.

1. A Differential-bigraded (DB) R-algebra is a homologically bigraded ring A with a graded ring homomomorphism from R to $A_{0,*}$, further equipped with a homological degree -1 map $\partial \colon A \to A$ satisfying $\partial^2 = 0$ and the *graded Leibniz rule*:

$$\partial(ab) = \partial(a)b + (-1)^{|a|}a\partial(b).$$

2. A differential-bigraded (DB) A-module is a bigraded A module M with a degree -1 map ∂_M satsfying $\partial_M^2 = 0$ and the *graded Leibniz rule*:

$$\partial(am) = \partial(a)m + (-1)^{|a|} a \partial_M(m).$$

3. A differential-bigraded (DB) ideal of A is a subset which is also a DB A-submodule.

4. A chain map of degree d from M to N is an R-linear map $f\colon M \to N$ of degree d which additionally satisfies that $\partial_N f = (-1)^d f \partial_M$.

5. When A is an DB R-algebra and B is a DB R-algebra equipped with a map $A \to B$ compatible with the structure maps of A and B in the sense that

$$\begin{array}{ccc} R & \longrightarrow & A \\ & \searrow & \downarrow \\ & & B \end{array}$$

commutes, then B is called a *DB A-algebra*.

6. A morphism of bigraded A-algebras is a bigraded ring homomorphism $B \to C$ which commutes with the A-algebra structure maps of B and C, and which is also a chain map.

Remark 2.3.3. When A is a DB R-algebra with differential ∂, R-linearity of ∂ follows from the Leibniz rule imposed by part (1) of the preceding definition, and the fact that R is concentrated in homological degree zero. Hence, forgetting multiplicative structure yields a chain complex of R-modules in the ordinary sense.

There is a functor from the category of $\mathbb{D}$-graded rings to the category of homological bigraded rings which sends R to the bigraded ring concentrated in homological degree zero. Using the 0 map as a differential then allows for R to be viewed as a DB algebra over itself, and for modules and algebras over R to be viewed as DB modules and algebras. Hence the categories of DB R algebras and DB R modules strictly enlarge the categories of $\mathbb{D}$-graded R algebras and $\mathbb{D}$-graded R-modules. When A is said to be a DB algebra without reference to a fixed ring R, this is understood to mean that A is a DB algebra over the $\mathbb{D}$-graded ring $A_{0,*}$.

In the other direction, there is a forgetful functor:

Definition 2.3.4. When M is a DB module over a DB algebra A, forgetting the differentials yields a bigraded module M over a bigraded algebra A. These are called the *underlying module of* M over the *underlying algebra of* A, and are denoted $M^\natural$ and $A^\natural$, respectively.

Definition 2.3.5. For a DB algebra A, set $Z(A) = \ker(\partial)$ and $B(A) = \operatorname{im}(\partial)$, the *cycles* and *boundaries* of A, respectively. The homology $H(A)$ is the subquotient $Z(A)/B(A)$.

The significance of the graded Leibniz rule is that the homology $H(A)$ of any DB R-algebra is bigraded R-algebra, which we now verify:

Proposition 2.3.6. *$Z(A)$ is a DB-subalgebra of A, $B(A)$ is a bihomogeneous ideal of $Z(A)$, and $H(A)$ is a bigraded R algebra.*

Proof. Since ∂ is an R-linear $\mathbb{D}$-graded map of homological degree -1, its image and kernel are bigraded R-modules. That $Z(A)$ is a subring follows from linearity of ∂ and the Leibniz rule, which gives

$$\partial(1) = \partial(1 \cdot 1) = 2\partial(1)$$

and so $1 \in \partial(A)$. That $B(A)$ is an ideal follows from linearity of ∂ and the Leibniz rule. $\square$

A number of standard procedures, such as direct sum and tensor product, are available for constructing DB algebras and modules and their properties are analogous for the similar constructions for complexes.

2.4 Adjoining Variables

An essential feature of DB-algebras is the ability to extend their differentials by a procedure colloqially refered to as "adjoining variables". The advantage is that given a DB-algebra A and a list of cycles, a differential may be defined on a (typically infinite rank) free A-module turning it into a DB A-algebra in which the chosen cycles are now bounderies. This allows the structure of infinite resolutions to be more concretely understood.

As the ring of interest R is always a DB algebra over itself, constructions typically proceed as a sequence of algebra extensions beginning at R. The constructions involve graded-commutative algebras possessing important freeness properties.

Definition 2.4.1. Let A be a bigraded R-algebra and x be an indeterminate over A of degree $(i, j) \in \mathbb{N} \times \mathbb{D}$ with i odd. The *exterior algebra on x* over A, denoted either by $A\langle x\rangle$ or $A[x]$ is the A-algebra

$$A \oplus Ax$$

with basis $\{1, x\}$, and A-bilinear multiplication defined by the multiplication table $x^2 = 0$, $x \cdot 1 = 1 \cdot x = x$. Using the sign of the A action on $Ax = \Sigma^i A$ as given in definition 2.2.5, this means that for $a, b, c, d \in A$, we have

$$(a + bx)(c + dx) = ac + (ad + (-1)^{|c|}bc)x.$$

The use of two notations $A\langle x\rangle$ and $A[x]$ to denote the exterior algebra anticipate its simultaneous role as the free strictly graded-commutative algebra and free strictly graded-commutative Γ-algebra on an odd homological degree variable, and the desire to have a single compact notation for the free strictly graded-commutative algebra and free strictly graded-commutative Γ-algebra on a set of variables of both even and odd homological degrees.

Definition 2.4.2. Let A be a bigraded R-algebra and x be an indeterminate over A of degree $(i, j) \in \mathbb{N} \times \mathbb{D}$ with i even. The *polynomial algebra on x* over A, denoted $A[x]$, is the A-algebra

$$A \oplus Ax \oplus Ax^2 \oplus \dots$$

with basis $\{x^n\}_{n\in\mathbb{N}}$ and A-bilinear multiplication defined by the multiplication table $x^n x^m = x^{n+m}$. By convention, $x^0 = 1$.

The $\mathbb{N}\times\mathbb{D}$-grading of the exterior and polynomial algebras is induced by the grading of A and the choice of bidegree (i, j), cf. [15, III, §7, Proposition 11] for exterior algebras and [15, III, § 6, Proposition 10] for polynomial algebras. The exterior and polynomial algebras assemble into the free strictly graded-commutative algebra on a set of variables, cf. [15, III, § 7, Proposition 1 and III, § 6, Proposition 2]:

Proposition 2.4.3. *For a bigraded set X, the directed system of subsets of X induces a directed system of algebras. The directed colimit $A[X]$ is the free strictly graded-commutative algebra on the set X, in the sense that it is adjoint to the forgetful functor from bigraded A-algebras to graded sets.*

Construction 2.4.4 (Exterior Variables)**.** Let A be a DB algebra, and let $z \in Z(A)$ be bihomogeneous with $|z|$ even. Let x be an indeterminate over A of degree $(|z|+1, \deg(z))$. Then the differential of A may be extended to $A\langle x\rangle$ by the assignment $\partial(x) = z$ and the imposition of the Leibniz rule, turning $A\langle x\rangle$ into a DB-algebra. To clearly specificy the differential of x, this DB algebra will be denoted by either $A\langle x \,|\, \partial(x) = z\rangle$ or $A[x \,|\, \partial(x) = z]$

Regardless of the status of $\operatorname{cls}(z)$ in $H(A)$, z is the boundary of x in $A\langle X\rangle$ and so its homology class is zero in $H(A\langle X\rangle)$, hence we say that one has *adjoined an exterior variable x to kill the cycle z.*

Construction 2.4.5 (Polynomial Variables)**.** Let A be a DB algebra, and let $z \in Z(A)$ be bihomogeneous with $|z|$ odd. Let x be an indeterminate over A of degree $(|z|+1, \deg(z))$. Then the differential of A may be extended to the polynomial algebra $A[x]$ by the assignment $\partial(x) = z$ and the imposition of the Leibniz rule, turning $A[x]$ into a DB-algebra. To clearly specify the differential of x, this DB algebra will be denoted by $A[x \,|\, \partial(x) = z]$.

Regardless of the status of $\operatorname{cls}(z)$ in $H(A)$, z is the boundary of x in $A\langle X\rangle$ and so its homology class is zero in $H(A\langle X\rangle)$, hence we say that one has *adjoined a polynomial variable x to kill the cycle z.*

2.5 Divided Powers

For the polynomial adjunction $A[x]$, a consequence of the Leibniz rule is the familiar "power rule" calculation $\partial(x^i) = izx^{i-1}$, valid for all i regardless of characteristic. Consequently x^p is a cycle when $\operatorname{char}(R) = p > 0$. With the goal of constructing algebra resolutions more economically, Tate [37] introduced into commutative algebra the use of the *free divided power algebra.*

Definition 2.5.1. Let A be a bigraded R-algebra and x be an indeterminate over A of degree (i, j) with i even. The *divided power algebra on* x over A, denoted $A\langle x\rangle$, is the A-algebra

$$A \oplus Ax \oplus Ax^{(2)} \oplus \dots$$

with basis $\{x^{(i)}\}_{i\in\mathbb{N}}$ and A-bilinear multiplication defined by the multiplication table $x^{(i)}x^{(j)} = \binom{i+j}{i}x^{(i+j)}$, where $\binom{i+j}{i}$ is the binomial coefficient. By convention, $x^{(1)}$ is denoted by x and $x^{(0)}$ by 1.

The consequence of this modification is that the equalities obtained by inductive application of the Leibniz rule will no longer involve integer coefficients other than 1. This avoids the introduction of additional cycles depending on the characteristic.

When a bigraded ring A is also a $\mathbb{Q}$-algebra, the family of functions $\gamma_n \colon A \to A$ given by $a \mapsto a^n/n!$ satisfy certain axioms. These axioms may be generalized to a family of maps, defined potentially on all of A, but more generally upon some specific ideal. Following [14], [22] and [19, A 2.4], we define this structure in its natural level of generality in the context of bigraded rings and algebras.

Definition 2.5.2. Let A be a bigraded R-algebra and $I \subset A$ be an ideal. The pair (A, I) is a *divided power ring* (or A has a *divided power structure on* I) if there exists a family of maps $\{\gamma_n \colon I \to I\}_{n\in\mathbb{N}}$ satisfying the following axioms for all $x, y \in I$, $a \in A$, and $n, m \in \mathbb{N}$:

1. $\gamma_0(a) = 1$ and $\gamma_1(a) = a$
2. $\deg(\gamma_n(a)) = n\deg(a)$ and $|\gamma_n(a)| = n|a|$
3. $\gamma_n(a)\gamma_m(a) = \binom{n+m}{n}\gamma_{n+m}(a)$
4. $\gamma_n(ax) = a^n\gamma_n(x)$

5. $\gamma_n(x+y) = \sum_{s+t=n} \gamma_s(x)\gamma_t(y)$ ("Freshman's Binomial Theorem")

6. $\gamma_n(\gamma_m(x)) = \frac{(nm)!}{n!(m!)^n}\gamma_{nm}(x)$

A general class of examples is provided by the following:

Proposition 2.5.3. *Let A be a bigraded (not necessarily strictly graded-commutative) ring, and let $I \subset A$ be an ideal satisfying $I^2 = 0$. Then there exists a divided power structure on I defined by $\gamma_0(a) = 1$, $\gamma_1(a) = a$, and $\gamma_n(a) = 0$ for $n \geq 2$.*

Proof. Axioms (1), (3), (4) and (6) all hold trivially. Axiom (2) holds trivially unless $n = m = 1$, for which the axiom holds since $x \in I$ implies $\gamma_1(x)\gamma_1(x) = x^2 = 0$. The condition $I^2 = 0$ gives that all interior terms in the expansion of $\gamma_n(x+y)$ are zero, verifying axiom (5). □

On the other hand, if $I^2 = 0$ and (A, I) is a divided power ring, it does not follow that the maps $\gamma_{\geq 2}$ are trivial. See [14, pp. 3.24–3.27] for more details.

Definition 2.5.4. Let (A, I) be a bigraded divided power ring. (A, I) is a *strictly graded commutative divided power ring* if A is a strictly graded-commutative bigraded ring and $\gamma_{\geq n}(a) = 0$ whenever $|a|$ is odd.

For the remainder of this work, all Γ-algebras will be assumed to be strictly graded-commutative.

Pairs of bigraded rings and bihomogeneous ideals (A, I) equipped with such a family of functions assemble into a category, called the *category of bigraded divided power rings*, in which the morphisms from (A, I) to (B, J) are bigraded maps from A to B which send I to J and commute with the divided power operators. It has the category of strictly graded-commutative bigraded divided power rings as a full subcategory.

Divided power rings may be combined with algebra structures.

Definition 2.5.5. Let (A, I) be a divided power ring. A *divided power A-algebra* or a Γ *A-algebra* is a divided power ring (B, J) equipped with a distinguished morphism of divided power rings from (A, I) to (B, J). A *morphism of divided power A-algebras* is a morphism of divided power rings which commutes with the structure maps of the algebras.

The following two examples are most frequently encountered.

Example 2.5.6. Let B be any A-algebra, and assume that B has a system of divided powers on an ideal I. A has a trivial system of divided powers on the 0-ideal. B is a Γ A-algebra with respect to this trivial system of divided powers on A. In particular, any bigraded divided power ring is always a divided power algebra over its homological degree 0 piece. Similarly, when C is any A-algebra and $\varphi\colon C \to B$ is any map of A-algebras, φ is a map of divided power algebras from $(C, (0))$ to (B, I).

Definition 2.5.7. Let (A, I) be any divided power ring. When $|x|$ is even, the divided power structure on A can be extended to a divided power structure on $A\langle x\rangle$ with respect to the ideal

$$IA\langle x\rangle + (x^{(\infty)}) = IA\langle x\rangle + (x, x^{(p)}, x^{(p^2)}, \dots)$$

by the assignments $\gamma_n\colon x \mapsto x^{(n)}$ and extended to arbitrary elements by imposing compatibility with the axioms. When $|x|$ is odd, the divided power structure on A can be extended to a divided power structure on $A\langle x\rangle$ with respect to the ideal

$$IA\langle x\rangle + (x)$$

by the assignments $\gamma_0(x) = 1$, $\gamma_1(x) = x$, and $\gamma_n(x) = 0$, extended to arbitrary elements by imposing compatibility with the axioms. These structures are the *divided power structure on* $A\langle x\rangle$ *induced by* (A, I).

As the prior example suggests, $A\langle x\rangle$ is the free (strictly graded-commutative) Γ-algebra over A on a variable x. When adjunction of variables is iterated to adjoin a (possibly infinite) set X, the result $A\langle X\rangle$ is still the free (strictly graded-commutative) Γ-algebra over A on the set X. The following may be found in [14], [19, A 2.4], or [22, Proposition 1.7.6].

Proposition 2.5.8. *Let* (A, I) *be a* Γ*-algebra, and* X *be a (possibly infinite) set of variables over* A*. Then* $A\langle X\rangle$ *is the* free strictly graded-commutative A-algebra with divided powers*, also called the free graded-commutative* Γ*-algebra, in that it is adjoint to the forgetful functor from the category of graded-commutative* Γ A*-algebras to the category of bigraded sets. In other words, when* B *is a bigraded* A*-algebra with divided powers on an ideal* $J \subset B$ *and* $\varphi\colon (A, I) \to B$ *is a morphism of divided power algebras, then each choice of set* $\{b_x\}_{x\in X}$ *with* $|b_x| = |x|$ *and* $\deg(b_x) = \deg(x)$ *determines a unique bigraded morphism* $\tilde{\varphi}\colon A\langle X\rangle \to B$ *satisfying* $\tilde{\varphi}(x) = b_x$ *and* $\tilde{\varphi}|_A = \varphi$*.*

Remark 2.5.9. Divided powers operations model the expressions $\frac{a^n}{n!}$ even when $n!$ is not invertible in R. In fact, when $\mathrm{char}(R) = 0$, A is any R-algebra, and x is a variable over A of even degree, the map $x^n \mapsto n!x^{(n)}$ is an isomorphism of algebras from $A[x]$ to $A\langle x\rangle$. In contrast, when $\mathrm{char}(R) = p > 0$, $x^p = p!x^{(p)} = 0$ in $A\langle x\rangle$, so the same map fails to be an isomorphism. More generally, when p is not invertible in R, $x^{(p^e)}$ is not contained in the ideal generated by x in $A\langle x\rangle$ for any $e \geq 1$. Hence to determine a DB-algebra structure on the divided power algebra requires the assignment of the differential of each of $x^{(p^e)}$. Alternatively, one may require that ∂ commute with the divided power operators by stipulating that, in addition to the Leibniz rule, ∂ respect the divided power operations in the sense that

$$\partial(\gamma_n(a)) = \partial(a)\gamma_{n-1}(a).$$

Maps satisfying these properties are called Γ-derivations, to be further discussed in definition 7.2.2.

Construction 2.5.10 (Divided Power Variables)**.** Let A be a DB algebra, and let $z \in Z(A)$ be bihomogeneous with $|z|$ odd. Let x be an indeterminate over A of degree $(|z|+1, \deg(z))$.

Then the differential of A may be extended to the divided power algebra $A\langle x\rangle$ by the assignments $\partial(x) = z$ and imposing that ∂ is a Γ-derivation: that it satisfies the Leibniz rule and $\partial(x^{(n)}) = \partial(x)x^{(n-1)}$. When necessary to explicitly specify the differential of x, this DB algebra will be denoted by $A[x|\partial(x) = z]$. Regardless of the status of cls(z) in $H(A)$, z is the boundary of x in $A\langle x\rangle$ and so its homology class is zero in $H(A\langle X\rangle)$, hence one says that one has *adjoined a divided power variable x to kill the cycle z.*

More generally, DB algebras can have a divided power structure even if there underlying algebra structure is not that of the free divided power algebra.

Definition 2.5.11. A DB A-algebra B has a *divided power structure on an ideal* $I \subset B$ if its underlying algebra $B^\natural$ has a divided power structure on $I^\natural$, and its differential is a Γ-derivation with respect to this structure, see definition 7.2.2. A map of DB algebras with divided powers $(B, I) \to (C, J)$ is a bigraded map of DB A-algebras $\varphi\colon B \to C$ which further satisfies $\varphi(I) \subset J$ and which commutes with the divided power operators.

2.6 Semifree Extensions

Classically, the polynomial algebra over a field k on a set of variables $x_1, \dots, x_n$ is denoted simply by $k[x_1, \dots, x_n]$. It may be obtained by sequentially adjoining a single variable at a time, in any of the $n!$-many orderings of the variables.

When the above constructions are iterated, giving sequences of adjunctions

$$A \hookrightarrow A\langle x_1 \rangle \hookrightarrow (A\langle x_1 \rangle)\langle x_2 \rangle \dots$$

and

$$A \hookrightarrow A[x_1] \hookrightarrow (A[x_1])[x_2] \dots,$$

for brevity we will similarly denote the result after n-steps by simply $A\langle X_{\leq n} \rangle$ and $A[X_{\leq n}]$. While compact, this notation has the disadvantage of obscuring the differential, and in particular, the fact that the differential of x_n may depend non-trivially on those previously adjoined. Consequently there may be fewer than $n!$-many ways of adjoining variables one at a time to reach a particular DB algebra. When multiple orderings are possible, the same DB algebra results. For example, the classical Koszul complex $R\langle x_1, \dots x_n \,|\, \partial(x_i) = r_i \rangle$ on a set of elements of R can be constructed by adjoining variables one at a time in any order (cf. [18, Proposition 1.6.6]). This concept extends to an arbitrary bigraded set X: $A[X]$ or $A\langle X \rangle$ will denote an algebra obtained by a sequence of adjunctions of variables according to some fixed, unspecified well-ordering of X.

Suppressing the differential has the disadvantage of suggesting that these constructions induce a functor $X \mapsto A[X]$ analogous to forming a polynomial algebra. This analogy only goes so far: the differential on X is generally non-trivial and involves elements of X adjoined at earlier stages, which can not be encoded in X alone (i.e., there is no functor $X \to A[X]$). Despite this flaw, these construction still possess some of the freeness properties of algebras without differential. To reflect this distinction, these constructions are called *semifree extensions*.

Definition 2.6.1. A DB algebra B is a *semifree extension of* A if $B \cong A[X]$ where $A[X]$ is obtained by a possibly infinite sequence of adjunctions of the form described in construction 2.4.4 and construction 2.4.5. For the rest of this work, $A[X]$ will imply that the set X has a differential unless specifically indicated otherwise.

Definition 2.6.2. A DB Γ-algebra (B, J) is a *semifree* Γ*-extension of a* Γ*-algebra* (A, I) if $(B, J) \cong (A\langle X\rangle$ where $A\langle X\rangle$ is obtained by a possibly infinite sequence of adjunctions of the form described in construction 2.4.4 and construction 2.5.10, with divided power structure on $A\langle X\rangle$ induced by the structure of (A, I). For the rest of this work, $A\langle X\rangle$ will imply that the set X has a differential unless specifically indicated otherwise.

Semifree extensions and semifree Γ-extensions enjoy more restricted lifting properties.

Proposition 2.6.3. *Let* $\varphi\colon A \to A'$ *be a morphism of DB algebras, and let* $\{z_\lambda\}_{\lambda\in\Lambda} = Z \subset Z(A)$ *satisfy that* $\varphi(Z) \subset B(A')$*, and* $\{x_\lambda\}_{\lambda\in\Lambda}$ *be a set of variables over* A*. Let* $\{b_\lambda\}_{\lambda\in\Lambda}$ *be any set satisfying* $\partial(b_\lambda) = z_\lambda$ *for all* $\lambda \in \Lambda$*. Then there is a lift* $\tilde{\varphi}\colon A[\{x_\lambda\} \,|\, \partial(x_\lambda) = z_\lambda] \to A'$ *of* φ *defined by sending* x_λ *to* b_λ*, furnished by the universal property of the free strictly graded-commutative algebraproposition 2.4.3. This lift is a bigraded morphism of DB-algebras.*

Proposition 2.6.4. *Let* $\varphi\colon (A, I) \to (A', I')$ *be a morphism of DB* Γ*-algebras, and* $\{z_\lambda\}_{\lambda\in\Lambda} = Z \subset Z(A)$ *satisfy that* $\varphi(Z) \subset B(A')$*, and* $\{x_\lambda\}_{\lambda\in\Lambda}$ *be a set of variables over* A*. Suppose there exists* $\{b_\lambda\}_{\lambda\in\Lambda} \subset I'$ *with* $\partial(b_\lambda) = z_\lambda$ *for all* $\lambda \in \Lambda$*. Then there is* *a lift* $\tilde{\varphi}\colon A\langle\{x_\lambda\} \mid \partial(x_\lambda) = z_\lambda\rangle \to A'$ *of* φ *defined by sending* x_λ *to* b_λ*, furnished by the universal property of the free strictly graded-commutative* Γ*-algebraproposition 2.5.8. This lift is a bigraded morphism of DB* Γ*-algebras.*

The proof of the following is essentially unaltered from that found in [8, Proposition 2.1.9], with the addition only of the internal grading.

Lemma 2.6.5 ([8], 2.1.9)**.** *If* $A[X]$ *is a semifree extension of a DB algebra* A*, then any diagram of morphisms of DB algebras over* A *represented by solid arrows*

$$\begin{array}{ccc} & & B \\ & \overset{\gamma}{\nearrow} & \simeq \downarrow {\scriptstyle \beta} \\ A[X] & \xrightarrow{\alpha} & C \end{array}$$

with a surjective quasi-isomorphism β *can be completed to a commutative diagram by a morphism* γ*, that is defined uniquely up to* A*-linear homotopy.*

Adjunction of variables results in controlled change in homology as shown by the following exact sequences. They are generalizations of the classical long exact sequence of Koszul homology, as found in e.g. [18, Proposition 1.6.12]. They are found in [37, Theorem 2], stated for general rings. The inclusion of grading requires that exact sequences incorporate a shift determined by the internal degree.

Proposition 2.6.6. *Let* $A\langle x \mid \partial(x) = z\rangle$ *be a semifree extension with* $|x| = i$ *odd and* $\deg(x) = j$*. There is an exact sequence*

$$0 \to A \xrightarrow{\alpha} A\langle x\rangle \xrightarrow{\beta} A(j) \to 0$$

where $\alpha(a) = a + 0x$ *is a homological degree zero map and* $\beta(a + bx) = b(j)$ *is a homological degree* i *map. The long exact sequence in homology is one of* $\mathbb{D}$*-graded* *modules in which the connecting homomorphism is multiplication by* cls(w)*:*

$$\begin{array}{l}\cdots \longrightarrow H_n(A) \xrightarrow{H_n(\alpha)} H_n(A\langle x\rangle) \xrightarrow{H_n(\beta)} H_{n-i}(A(j)) \xrightarrow{\mathrm{cls}(z)} \\ \hookrightarrow H_{n-1}(A) \xrightarrow{H_{n-1}(\alpha)} H_{n-1}(A\langle x\rangle) \xrightarrow{H_{n-1}(\beta)} H_{n-1-i}(A(j)) \longrightarrow \cdots\end{array}$$

Proposition 2.6.7. *Let* $A\langle x \,|\, \partial(x) = z\rangle$ *be a semifree extension with* $|x| = i$ *even and* $\deg(x) = j$*. There is an exact sequence*

$$0 \to A \xrightarrow{\alpha} A\langle x\rangle \xrightarrow{\beta} A\langle x\rangle(j) \to 0$$

where $\alpha(a) = a + 0x$ *is a homological degree zero map and*

$$\beta(\sum a_l x^{(l)}) = \sum a_l x^{(l-1)}$$

is a homological degree i *map. The long exact sequence in homology is one of* $\mathbb{D}$*-graded modules:*

$$\begin{array}{l}\cdots \longrightarrow H_n(A) \xrightarrow{H_n(\alpha)} H_n(A\langle x\rangle) \xrightarrow{H_n(\beta)} H_{n-i}(A\langle x\rangle(j)) \longrightarrow \\ \hookrightarrow H_{n-1}(A) \xrightarrow{H_{n-1}(\alpha)} H_{n-1}(A\langle x\rangle) \xrightarrow{H_{n-1}(\beta)} H_{n-1-i}(A\langle x\rangle(j)) \longrightarrow \cdots\end{array}$$

Unlike in proposition 2.6.6, the connecting homomorphism in proposition 2.6.7 is not multiplication by cls(w). For a more detailed discussion see [4, Remark 6.1.6].

These sequences demonstrate the utility of divided powers in positive characteristic, as the maps in proposition 2.6.7 are chain maps precisely because of the divided power structure. The exact sequence of proposition 2.6.7 can still be used to provide some information about adjoining polynomial variables, since their initial segments are the same. Since $A\langle x \,|\, \partial(x) = z\rangle$ is a strictly graded-commutative algebra, proposition 2.6.3 furnishes a comparison map $A[x\,|\partial(x) = z] \to A\langle x\rangle$. It sends x^n to $n!x^{(n)}$, and so is invertible in all those homological degrees for which $n!$ is invertible. We make this more precise below:

Proposition 2.6.8. *When $A_{0,0}$ is a ring of equicharacteristic zero, $A[x] \cong A\langle x\rangle$ and so $A[x]$ may be substituted for $A\langle x\rangle$ in the long exact sequence of proposition 2.6.7. When $A_{0,0}$ is a ring of characteristic $p > 0$ or a local ring of mixed characteristic $p > 0$, then there is an isomorphism of truncated chain complexes*

$$A[X]_{\leq |x|p-1} \cong A\langle X\rangle_{\leq |x|p-1}$$

and so $A[x]$ may be substituted for $A\langle x\rangle$ in the following truncation of the exact sequence of proposition 2.6.7:

$$H_{|x|p-1}(A) \xrightarrow{H_{|x|p-1}(\alpha)} H_{|x|p-1}(A[x]) \xrightarrow{H_{|x|p-1}(\beta)} H_{|x|p-1-i}(A[x](j)) \to$$
$$\hookrightarrow H_{|x|p-2}(A) \xrightarrow{H_{|x|p-2}(\alpha)} H_{|x|p-2}(A[x]) \xrightarrow{H_{|x|p-2}(\beta)} H_{|x|p-2-i}(A[x](j)) \longrightarrow \dots$$

The change in homology is most well-understood in the case of regular elements, which we now define. A subtle change is necessary from the classical definition to account for elements of odd degree squaring to zero.

Definition 2.6.9. Let A be a bigraded ring. When $a \in A$ is of even homological degree, a is regular if $ab = 0$ implies $b = 0$, i.e., that $(0 : a) = 0$. When $a \in A$ is of odd homological degree, a is regular if $ab = 0$ implies $b = ca$, i.e., that $(0 : a) = (a)$.

Remark 2.6.10. Examining the homology exact sequences of proposition 2.6.6, proposition 2.6.7, and proposition 2.6.8 in low homological degrees shows that when $|z| = n$

and $i \leq n$,

$$H_i(A[x \,|\, \partial(x) = z]) \cong H_i(A\langle x \,|\, \partial(x) = z\rangle \cong \begin{cases} H_i(A) & i < n \\ H_n(A)/\operatorname{cls}(z)H_n(A) & i = n \end{cases}.$$

This justifies the use of the phrase "adjoining x to kill the cycle z". See [22, Lemma 1.2.1] and [8, Construction 2.1.8] for more details. Since the connecting homomorphism in proposition 2.6.6 is multiplication by $\operatorname{cls}(z)$, when $\operatorname{cls}(z)$ is regular in $H(A)$ and of even degree, then the connecting homomorphism is injective, so the long exact sequence breaks up into short exact sequences. Consequently, when $\operatorname{cls}(z) \in H(A)$ is regular, $H(A\langle x\rangle) \cong H(A)/\operatorname{cls}(z)H(A)$. Using a more complicated spectral sequence argument, the same statement can be made when $\operatorname{cls}(z)$ is a regular element of odd degree. The proof found in Avramov [8, Proposition 6.1.7] holds for arbitrary algebras and so holds in the $\mathbb{D}$-graded case without modification.

2.7 Direct Colimits of DB algebras

The construction of a semifree extension (or Γ-extension) by adjoining variables one at a time is an example of a general phenomenon, often referred to in commutative algebra as a directed limit. We will adopt the naming convention from category theory, which instead uses the term "directed colimit"[33, Chapter 3]. When the indexing set is the natural numbers with the ordinary ordering, we will sometimes use the term "sequential colimit".

By definition, a semifree extension is always obtainable as a directed colimit of a sequence of semifree extensions in which, at each step, a single variable is adjoined. On the other hand, the colimit of an arbitrary directed system of semifree extensions need not itself be semifree. In fact, the directed colimit of free strictly graded-commutative algebras need not be free, as the following example shows:

Example 2.7.1. Let $|x| = \deg(x) = 0$, and let $f\colon k[x] \to k[x]$ be the k-algebra map defined by sending x to x^2. Consider the diagram

$$k[x] \xrightarrow{f} k[x] \xrightarrow{f} k[x] \xrightarrow{f} \dots$$

The images of the x's in the sequential colimit are non-units which have all 2^i'th roots for every i, which is not possibly in a polynomial ring over any field.

The following lemmas show the needed criteria for the directed colimit of semifree extensions, indexed over the natural numbers, to again be a semifree extension. First, it is necessary that the colimit of the underlying algebras be a free bigraded algebra. The next lemma shows that this holds provided that in each fixed homological degree, the variables are eventually mapped to variables.

Lemma 2.7.2. *Let A be a bigraded algebra and let*

$$\{A[X^1] \xrightarrow{f} A[X^2] \xrightarrow{f} \dots\}$$

be a sequence of free bigraded algebras over A. Suppose that for each $l \in \mathbb{N}$, there exists $i_l \gg 0$ such that $f^{j-i}(X^i_l) \subset X^j_l$ for all $j > i > i_l$. Then there is a bigraded A algebra $A[X]$ and maps $f_i : A[X^i] \to A[X]$ so that $(A[X], \{f_i\})$ is the directed colimit.

Proof. Let $\mathbb{N}$ be equipped with the natural order and $N = \sqcup_{l\in\mathbb{N}}\mathbb{N}_l$ be equipped with the disjoint union ordering (each $\mathbb{N}_l$ is just a copy of $\mathbb{N}$). Then $\operatorname{colim}_{\mathbb{N}}(A[X^i]) \cong \operatorname{colim}_N(A[X^i_l])$, since

$$\operatorname{colim}_{\mathbb{N}}(A[X^i]) \cong \operatorname{colim}_{\mathbb{N}}(\otimes_{l\in\mathbb{N}} A[X^i_l]) \cong \operatorname{colim}_{\mathbb{N}}\left(\coprod_{l\in\mathbb{N}} A[X^i_l]\right) \cong \operatorname{colim}_N(A[X^i_l])$$

Set $L = \sqcup_{l\in\mathbb{N}}\{i > i_l\}$. Then L is cofinal to the poset N. Hence $\mathrm{colim}_N(A[X_l^i]) \cong \mathrm{colim}_L(A[X_l^i])$.

By the assumptions on i_l, restriction of f to X_l^i induces a directed system of bigraded sets; let X be its colimit. The free strictly bigraded-commutative algebra functor is cocontinuous (preserves colimits), so

$$\mathrm{colim}_L\, A[X_l^i] \cong A[\mathrm{colim}_L\, X_l^i] \cong A[X]$$

which is a free bigraded algebra, as desired. □

The above lemma only deals with underlying algebra structure. The next lemma shows that directed colimits of DB algebras can be computed in the category of algebras (i.e., after forgetting differentials).

Lemma 2.7.3. *Let $\{(B^i, f^{ij})\}_{i\leq j\in I}$ be a directed system of DB A-algebras and let $(B, \{f^i\colon A^i \to B\}_{i\in I})$ be the directed colimit. Then $B^\natural$ is the directed colimit of the directed system of underlying algebras $\{(B^i)^\natural\}$.*

Proof. We prove the statement in reverse: starting with $B^\natural$ the directed colimit of the underlying algebras, we show that a differential may be defined on $B^\natural$ realizing it as the directed colimit.

for each $a \in B^\natural$, obtain some $i \in I$ and $b \in (B^i)^\natural$ such that $f^i(b) = a$, and define $\partial(a) = f^i(\partial(b))$. We show that that this is a well defined differential turning $B^\natural$ into a DB-algebra B: let $j \in \mathbb{N}$ and $c \in B^j$ also satisfy $f^j(c) = a$. Since I is directed, there exists $k \in I$ with $k \geq i$ and $k \geq j$. By the definition of the directed colimit, $f^k(f^{ik}(b)) = a$ and $f^k(f^{jk}(c)) = a$. Since f^{ik} and f^{jk} are chain maps and by the definition of the directed colimit, we have

$$f^i(\partial(b)) = f^k(f^{ik}(\partial(b))) = f^k(\partial(f^{ik}(b))).$$

Similarly, $f^j(\partial(b)) = f^k(\partial(f^{jk}(b)))$. Hence it suffices to show the result for $i = j = k$.

Since $f^k(b-c) = a - a = 0$, there exists some $l \geq k$ such that $f^{kl}(b-c) = 0$. Then since f^{kl} is a chain map, $f^{kl}(\partial(b-c)) = 0$. Hence $f^l(f^{kl}(\partial(b-c))) = 0$, and by the definition of the directed colimit, $f^k(\partial(b-c)) = 0$. Hence $f^k(\partial(b)) = f^k(\partial(c))$, and so the differential on B is well-defined.

Equipped with this differential, each $f^i : B^i \to B$ is a chain map. For any DB A-algebra C with chain maps $g^i : B^i \to C$ for each i, there is a unique map of underlying algebras $\eta : B^\natural \to C^\natural$ making all of the triangles commute, since B was taken to be the colimit of the underlying algebras. For each $a \in B$, obtain some $i \in I$ and $b \in B^i$ such that $f^i(b) = a$. Then since g^i is a chain map,

$$\partial\eta(a) = \partial g^i(b) = g^i\partial(b) = \eta\partial(a).$$

Hence η is a chain map. Therefore B equipped with the differential above is the colimit in the category of differential graded algebras. $\square$

Chapter 3

Minimality and Canonical Resolutions

3.1 Acyclic Closures and Minimal Models

Our ultimate aim is to build DB algebra resolutions of quotient rings (and more generally of homomorphisms). In order that the structure of the resolution efficiently encode the information of the resolved object, it will be necessary to introduce a notion of minimality.

In the classical case of forming a minimal resolution of an R-module (without an algebra structure on the resolution), the construction proceeds inductively according to the following procedure.

1. First choose a minimal generating set of M, which defines a surjection $F_0 \twoheadrightarrow M$ with F_0 a free R-module

2. On step i, choose a minimal generating set for the kernel of the map from F_{i-1} to F_{i-2}, and use this generating set to define a map from a free module F_i to F_{i-1}

To construct DB algebra resolutions this approach will be emulated, but with adjunction of variables substituted for choosing bases of free modules. When R is a local ring, "choosing minimal generating sets" is made formal by use of bases of

$M/\mathfrak{m}_R M$, where $\mathfrak{m}_R$ is the maximal ideal of R. In the graded case, a suitable analogue holds for $\mathbb{D}$-local rings by lemma 2.1.8.

For the remainder of this chapter, R will be a commutative Noetherian $\mathbb{D}$-graded ring, where $\mathbb{D}$ is a commutative cancelative monoid with identity element denoted by 0.

When A is a DB algebra over a $\mathbb{D}$-local ring R, $A^\natural$ has its own irrelevant ideal with respect to the $\mathbb{N} \times \mathbb{D}$-grading. To include the differential structure, we require that this ideal be closed under the differential.

Definition 3.1.1. A DB R-algebra A is $\mathbb{D}$-*local* if the ideal generated by all bihomogeneous non-units of A is proper and closed under the differential, and the structure map $R \to A$ maps $\mathfrak{m}_R$ to this ideal. In this case, the ideal is denoted by $\mathfrak{m}_A$.

Proposition 3.1.2. *Let A be a $\mathbb{D}$-local DB-algebra. Then A_0 is a $\mathbb{D}$-local ring, and $\mathfrak{m}_A = \mathfrak{m}_{A_0} + A_{\geq 1}$.*

Proof. For $a, b \in A$, $ab = 1$ implies $|a| + |b| = 0$, and hence the bihomogeneous elements of $A_{\geq 1}$ are all non-units. Therefore for $a \in A_0$, a is a unit in A_0 if and only if it is a unit in A. □

The condition on the structure map is necessary to avoid situations like the following:

Example 3.1.3. Let $R = k[x]$ be $\mathbb{Z}$-graded by $\deg(x) = 1$. Interpret $A = k[x, x^{-1}]$ as a DB R-algebra with differential zero. $\mathfrak{m}_R = (x)$ and $\mathfrak{m}_A = (0)$, and the structure map is injective, so A is not a $\mathbb{Z}$-local R-algebra, even though it is $\mathbb{Z}$-local as a ring.

We now discuss two canonical resolutions. They differ only in positive characteristic, due to the use of divided power versus polynomial variables.

Definition 3.1.4. An *augmented DB R-algebra* is a DB R-algebra A equipped with a surjective A-linear bigraded R-algebra homomorphism from A to a quotient ring S of R (where S is concentrated in homological degree 0). It is *connected* if $A_0 = R$.

The following construction is classical when A is an augmented DG algebra over a local ring R, φ is the augmentation map, and $B = S$ is the target of the augmentation. Its proof may be found in many sources, including [37], [22], and [8]. We are unaware of a proof of the general case, in which R is $\mathbb{D}$-local and B is no longer concentrated in homological degree 0, so we include it below.

Theorem 3.1.5. *Let $\varphi\colon (A, I) \twoheadrightarrow (B, B_{\geq 1})$ be a bigraded surjective morphism of $\mathbb{D}$-local DB Γ R-algebras, and suppose that for each i, $H_i(A)$ and $H_i(B)$ are finitely generated R-modules. Set $K = \ker(H(\varphi))$. Then there exists a commutative diagram*

$$\begin{array}{ccccc} & & A\langle X\rangle & & \\ & {\scriptstyle\iota}\nearrow & & \searrow{\scriptstyle\widetilde{\varphi}}\ \simeq & \\ A & & \xrightarrow{\quad\varphi\quad} & & B \end{array}$$

satisfying the following conditions:

1. *$\iota\colon A \hookrightarrow A\langle X\rangle$ is a semifree Γ-extension*
2. *$\widetilde{\varphi}$ is a quasi-isomorphism*
3. *$X = X_{\geq 1, J}$ where $J = \{d + \deg(a) \,|\, d \in \mathbb{D},\, a \in K\}$ is the ideal in $\mathbb{D}$ generated by $\{\deg(a) \,|\, a \in K\}$*
4. *For each $i \in \mathbb{N}$, let $\widetilde{\varphi}_{(i)} = \widetilde{\varphi}|_{A\langle X_{\leq i}\rangle}$. Then $\{\mathrm{cls}(\partial(x)) \,|\, x \in X_i\}$ is a minimal set of homogeneous generators for $\ker(H_{i-1}(\widetilde{\varphi}_{(i-1)}))$.*

Proof. Set $\varphi_{(0)} = \varphi$, and $\iota_{(0)} = \mathrm{id}_A$. We prove the claim by induction. As a base case, if φ is already a quasi-isomorphism, then the trivial semifree Γ-extension with

$X = \emptyset$ satisfies all the required conditions. Let $i \geq 1$ Suppose for induction that a factorization

$$\begin{array}{ccccc} & & A\langle X_{\leq i-1}\rangle & & \\ & \overset{\iota_{(i-1)}}{\nearrow} & & \overset{\widetilde{\varphi}_{(i-1)}}{\searrow} & \\ A & & \xrightarrow{\quad\varphi\quad} & & B \end{array}$$

exists satisfying

1. $\iota_{(i-1)}$ is a semifree Γ-extension

2. $H_j(\widetilde{\varphi}_{(i-1)})$ is an isomorphism whenever $j < i$

3. $X_{\leq i-1}$ is concentrated in positive homological degrees and in internal degrees coming from the ideal J

4. For each $1 < j < i$, $\{\mathrm{cls}(\partial(x)) \,|\, x \in X_j\}$ is a minimal set of homogeneous generators for $\ker(H_{j-1}(\widetilde{\varphi}_{(j-1)}))$.

5. $H_{i-1}(\widetilde{\varphi}_{(i-1)})$ is surjective

6. $\ker(H_{i-1}(\widetilde{\varphi}_{(i-1)}))$ is finitely generated

Let $\mathrm{cls}(a_1), \ldots, \mathrm{cls}(a_n)$ be a minimal set of homogeneous generators of the $\mathbb{D}$-graded R module $\ker(H_{i-1}(\widetilde{\varphi}_{(i-1)}))$. Writing each $a_l = a + b$ with $a \in A$ and b in the ideal generated by $\{x^{(n)} \,|\, x \in X_{\leq i-1},\, n \in \mathbb{N}\}$, by condition (3) $\deg(b) \in J$ which forces $\deg(a_l) \in J$ unless $b = 0$. In this case, $a_l \in A$ belongs to $\ker(H_i(\varphi))$ and so $\deg(a) \in J$ by definition of J.

By choice of $a_1, \ldots, a_n$, $\mathrm{cls}(\widetilde{\varphi}_{(i-1)}(a_j)) = 0$ for each j, so we can choose elements $b_1, \ldots, b_n \in B_i$ which satisfy $\partial(b_j) = \widetilde{\varphi}_{i-1}(a_j)$ for each j. Let $X_i = \{x_1, \ldots, x_n\}$ be a set of variables over $A\langle X_{\leq i-1}\rangle$, and set

$$A\langle X_{\leq i}\rangle = (A\langle X_{\leq i-1}\rangle)\langle X_i\rangle = (A\langle X_{\leq i-1}\rangle)\langle x_1, \ldots, x_n \,|\, \partial(x_j) = b_j\rangle$$

The composition $\iota_{(i)}$ from A to $A\langle X_{\leq i-1}\rangle$ to $A\langle X_{\leq i}\rangle$ defines a semifree extension of A. Since B has a divided power structure on all elements of positive homological degree, proposition 2.6.4 furnishes a lift $\widetilde{\varphi_{(i)}}$ of $\widetilde{\varphi_{(i-1)}}$, and hence of φ. By repeated application of remark 2.6.10,

$$H_i(A\langle X_{\leq i}\rangle) \cong H_i(A\langle X_{\leq i-1}\rangle)/\ker(H_i(\widetilde{\varphi}_{(i-1)})) \cong H_i(B)$$

and $H_j(A\langle X_{\leq i}\rangle) \cong H_j(B)$ whenever $1 < j < i$, hence condition (2) is satisfied.

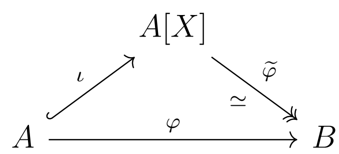
The bigraded module $\ker(H_i(\widetilde{\varphi}_{(i)}))$ is finitely generated over A_0 by proposition 2.6.6 and proposition 2.6.7, since R is Noetherian, and since finite generation satisfies the two-out-of-three property on exact sequences. Furthermore, $H_i(\varphi_{(i)})$ can be seen to be surjective by examining the long exact sequence in homology induced by $0 \to \ker(\varphi_{(i)} \to A\langle X_{\leq i}\rangle \xrightarrow{\varphi_{(i)}} B \to 0$ and using that $H_j(A\langle X_{\leq i-1}\rangle \cong H_j(B)$ for $j < i$.

Hence we have constructed a diagram satisfying all the required properties of the induction hypothesis. The sequential limit $A\langle X\rangle = \bigcup_i A\langle X_{\leq i}\rangle$ satisfies all the required properties (1)-(4) of the theorem. □

Definition 3.1.6. A DB algebra $A\langle X\rangle$ satisfying the conclusion of theorem 3.1.5 is called an *acylic closure* of φ. If $A \to S$ is an augmented R-algebra, then S has a trivial system of divided powers on elements of positive homological degree. Hence an acyclic closure of the augmentation to S always exists. An acyclic closure of the augmentation map is called an *acyclic closure of S over A*.

The acyclic closure construction can be mimicked but with polynomial rather than divided power variables. The proof of existence is the same except without concern for divided powers, and we omit it.

Theorem 3.1.7. *Let $\varphi\colon A \twoheadrightarrow B$ be a bigraded surjective morphism of DB R-algebras, and suppose that for each i, $H_i(A)$ and $H_i(B)$ is a finitely generated R-module. Set $K = \ker(H(\varphi))$. Then there exists a commutative diagram*



satisfying the following conditions:

1. *$\iota\colon A \hookrightarrow A[X]$ is a semifree extension*

2. *$X = X_{\geq 1,J}$ where J is the ideal $\{d + \deg(a) \,|\, d \in \mathbb{D},\ a \in K\}$ in $\mathbb{D}$ generated by $\{\deg(a) \,|\, a \in K\}$*

3. *$\{\mathrm{cls}(\partial(x)) \,|\, x \in X_1\}$ is a minimal set of homogeneous generators for $K/\partial(A_1)$*

4. *for $i > 1$, $\{\mathrm{cls}(\partial(x)) \,|\, x \in X_i\}$ is a minimal set of homogeneous generators for $H_{i-1}(A\langle X_{\leq i-1})$*

Definition 3.1.8. A DB algebra $A[X]$ satisfying the conclusion of theorem 3.1.7 is called a *minimal model* of φ. If $A \to S$ is an augmented R-algebra, then a minimal model of the augmentation map is called a *minimal model of S over A*.

In equicharacteristic 0, the minimal model and acyclic closure of a map are the same: by inductive application of proposition 2.6.8 each step in the construction of the minimal model and acyclic closure coincide. Otherwise, they can be quite different.

Example 3.1.9. Let $R = k[a]/(a^2)$ and k be a field of characteristic $p > 0$. The minimal resolution of k is

$$\ldots \xrightarrow{a} R \xrightarrow{a} R \xrightarrow{a} R \to 0$$

which is of the form $R\langle x_1, x_2 \,|\, \partial(x_1) = a,\, \partial(x_2) = ax_1\rangle$. This is the acyclic closure of the augmentation $R \to k$. The corresponding semifree extension $R[x_1, x_2]$ is not acyclic: $\partial(R[x_1, x_2]_{2p+1}) = aR[x_1, x_2]_{2p}$ which is a proper submodule since a is not invertible, but $\partial(x_2^p) = 0$ so $\partial(R[x_1, x_2]_{2p}) = 0$.

3.2 Minimality Conditions

Neither acyclic closures nor minimal models need be minimal free resolutions in the ordinary module-theoretic sense.

Example 3.2.1. Let k be a field of characteristic 0 and let $Q = k[a, b]$ be augmented by the natural surjection to $k[a, b]/(a^2, ab)$. The acyclic closure and minimal model both begin with the semifree extension

$$R[x_1, x_2, x_3, x_4 \,|\, \partial(x_1) = a^2,\, \partial(x_2) = ab,\, \partial(x_3) = bx_1 - ax_2, \partial(x_4) = ax_3 - x_1x_2]$$

which is not minimal as a complex since $ax_3 - x_1x_2 \notin \mathfrak{m}_R R[x_1, x_2, x_3, x_4]$.

Despite this failure, acyclic closures and minimal models both still enjoy minimality properties reminiscent of the property $\partial(F) \subset \mathfrak{m}F$. To keep these distinctions clear, we introduce the following terminology:

Definition 3.2.2. Let R be a $\mathbb{D}$-local ring with homogeneous maximal ideal $\mathfrak{m}$. A complex F of free R-modules is *minimal* if $\partial(F) \subset \mathfrak{m}F$. A DB algebra A is *minimal as a complex* if its underlying complex is one of free R-modules and is minimal.

The additional structure of the internal grading of R allows for minimality of complexes to be checked in particular internal degrees, as well as particular homological degrees.

Definition 3.2.3. Let R be a $\mathbb{D}$-local ring with homogeneous maximal ideal $\mathfrak{m}$. Let $i \in \mathbb{Z}$ and $N \subset \mathbb{Z}$, and let $d \in G(\mathbb{D})$ and $D \subset G(\mathbb{D})$. A chain complex F of free modules over R is *minimal in homological degree i and internal degree d* if $\partial(F_{i,d}) \subset \mathfrak{m}F$. F is *minimal in all N-homological degrees and all D-internal degrees* if $\partial(F_{N,D}) \subset \mathfrak{m}F$.

The following ideals and complexes associated to extensions will be used to describe alternative notions of minimality. When A is a $\mathbb{D}$-local R-algebra, there is an associated filtration $A \supseteq \mathfrak{m}_A \supseteq \mathfrak{m}_A^2 \subseteq \dots$ by powers of the maximal ideal. For divided power algebras an alternative filtration is needed.

Definition 3.2.4. A be a $\mathbb{D}$-local DB R-algebra with $A_0 = R$, and suppose that A has a system of divided powers on the ideal $A_{\geq 1}$ of all elements of positive degree. For each n, the n-divided power ideal of $\mathfrak{m}_A$, denoted $\mathfrak{m}_A^{(n)}$, is the ideal generated by all products of the form $r^{n_0}(a_1)^{(n_1)} \cdots (a_l)^{(n_l)}$, where $r \in \mathfrak{m}_R$, $a_j \in A_{\geq 1}$ for all j, and $n_0 + \cdots + n_l \geq n$. Then A has the filtration $A \supseteq \mathfrak{m}_A \supseteq \mathfrak{m}_A^{(2)} \supseteq \mathfrak{m}_A^{(3)} \dots$.

Example 3.2.5. Let k be a field of characteristic 2, and let $k\langle x\rangle$ be a semifree Γ-extension with $|x| = 2$. Then $(x)^{(2)} = \mathrm{Span}_k\{x^{(2)}, x^{(3)}, \dots\}$. The ideal generated by $x^{(2)}$ does not contain $x^{(4)}$, since $x^{(2)}x^{(2)} = 0$, and so the ideals $(x), (x^{(2)}), (x^{(3)}), \dots$ fail to form a filtration of $k\langle x\rangle$. This example explains the use of $\geq$ in definition 3.2.4.

Proposition 3.2.6. *Let A be a $\mathbb{D}$-local R-algebra. Then $\mathfrak{m}_A^n$ is a DB ideal for each n. If $(A, \mathfrak{m}_A)$ is a DB Γ-algebra, then $\mathfrak{m}_A^{(\geq n)}$ is a DB ideal for each n.*

These ideals can be used to define important quotient complexes of algebras.

Definition 3.2.7. Let A be a $\mathbb{D}$-local DB R-algebra. The *indecomposable complex of A*, denoted $\operatorname{ind} A$, is the quotient complex $\mathfrak{m}_A/\mathfrak{m}_A^2$. It is always a complex of $A_0/\mathfrak{m}_{A_0}$-modules, and so is a complex of $\mathbb{D}$-fields. In particular, when $A_0 = R$, it is a complex of $R/\mathfrak{m}_R$-modules. If A has a divided power structure on the ideal $\mathfrak{m}_A$, then $\mathfrak{m}_A/(\mathfrak{m}_A)^{(\geq 2)}$ is the Γ-*indecomposable complex* $\operatorname{ind}^\gamma A$ of A.

These complexes allow for notions of minimality to be defined. This coincides with the notion of minimality discussed in [8, Ch. 7] and [13, Sec. 1] for semifree extensions of regular rings.

Definition 3.2.8. Let A be a $\mathbb{D}$-local DB algebra over $R = A_0$. A is *absolutely minimal* if either of the following hold:

1. $\partial(\mathfrak{m}_A) \subset \mathfrak{m}_A^2$
2. $\operatorname{ind} A$ is minimal as a complex of $R/\mathfrak{m}_R$-modules (i.e., its differential is zero).

An alternative notion is available for algebras with divided powers.

Definition 3.2.9. Let R be $\mathbb{D}$-local, and let A be a DB A-algebra with a system of divided power on $A_{\geq 1}$. A is *absolutely* Γ-*minimal* if any of the following hold:

1. $\partial(\mathfrak{m}_A) \subset \mathfrak{m}_A^{(\geq 2)}$
2. $\operatorname{ind}^\gamma A$ is minimal as a complex of $R/\mathfrak{m}_R$-modules (i.e., its differential is zero).

Both notions are necessary, as the following example illustrates:

Example 3.2.10. Let k be a field of characteristic $p > 0$, let $|x| = i > 0$ be even, and consider the semifree Γ-extension of k given by $k\langle x, y \,|\, \partial(x) = 0, \partial(y) = x^{(p)}\rangle$. Then $x^{(p)}$ is not contained in the ideal $(x, y)^2$, So in degrees ip and $ip + 1$, the complex $\operatorname{ind} k\langle x, y\rangle$ has the form

$$0 \to ky \xrightarrow{\cong} kx^{(p)} \to 0$$

and so is not minimal. The complex $\operatorname{ind}^{\gamma} k\langle x\rangle$ is the k space $kx \oplus ky$ with both x and y mapping to zero, and so is minimal. Hence $k\langle x, y\rangle$ is absolutely Γ-minimal but not absolutely minimal.

Minimality has also been studied in the context of extensions of augmented algebras, cf. [22, §9], [8, §6]. In this context, minimality is not an absolute property of the algebra itself, but is a *relative property*, depending on how the algebra relates to the target of the augmentation. We define these notions in the more general context that minimal models and acyclic closures have been defined in this work. Various treatments of minimality in previous literature have subtle differences, and will seek to clarify the situation by introducing new terminology.

First, we need additional notation. The variables in semifree extensions allow for additional word-length grading to be introduced. This allows for the filtrations introduced for definition 3.2.8 and definition 3.2.9 to be replaced by filtrations based on the set of variables.

Definition 3.2.11. Let $A \hookrightarrow A[X]$ be a semifree extension.

1. For each n, the words of length n of the extension, denoted X^n, is the set of products of the elements of X which involve exactly n-many factors.

2. (X^n) is the ideal generated by X^n, and AX^n is the A-linear span of X^n (note that it need not be closed under the differential).

3. $A[X]$ may be given a new $\mathbb{N}$ grading by the assignments $A[X]_n = AX^n$. This is the *word-length grading* of the extension. Note that A sits in degree 0 with respect to this grading.

Definition 3.2.12. Let $A \hookrightarrow A\langle X\rangle$ be a semifree Γ-extension. For each n, the words of Γ-length n of the extension, denoted $X^{(n)}$, is the set

$$\{x_1^{(n_1)} \cdots x_{n_l}^{(n_l)} \,|\, n_1 + \cdots + n_l = n\}.$$

The *ideal of Γ-word-length at least* n, denoted $(X^{(\geq n)})$, is the ideal generated by $X^{(n)} \cup X^{(n+1)} \cup \ldots$. As an A-algebra, $A\langle X\rangle$ is $\mathbb{N}$-graded by $A\langle X\rangle_n = AX^{(n)}$, this is the *word-length grading* of the extension.

The following complexes allow for minimality to be identified in a relative context.

Proposition 3.2.13. *Let $A[X]$ be a $\mathbb{D}$-local semifree extension of A. Then $A + \mathfrak{m}_A(X) + (X^2)$ is a DB A-module. Now suppose that $\varphi\colon A \to S$ is augmented, and that the augmentation lifts to an augmentation $\widetilde{\varphi}\colon A[X] \to S$, and set $K = \ker(\varphi)$. Then $A + K(X) + (X^2)$ of $A[X]$ is a DB A-module.*

Remark 3.2.14. The use of A in the expression $A + \mathfrak{m}_A(X) + (X^2)$ refers to the subalgebra $A \subset A[X]$, not the ideal generated by A with $A[X]$. (X) and (X^2) in these expressions are ideals of $A[X]$, but not closed under the differential. Only $\mathfrak{m}_A$ and K are DB ideals of $A[X]$.

Proof. The underlying module of each term in $A + \mathfrak{m}_A(X) + (X^2)$ is evidently an $A^\natural$-module, so we only need verify that it is closed under the differential. This can be verified by checking each term of the sum:

1. If $a \in A$, then $\partial(a) \subset A \subset A + \mathfrak{m}_A(X) + (X^2)$ because $A \subset A[X]$ is a DB subalgebra.

2. For $a \in \mathfrak{m}_A$ and $b \in (X)$, $\partial(ab) = \partial(a)b \pm a\partial(b)$. Writing $\partial(b) = \sum_{l \in \mathbb{N}} b_l$ as a sum of homogeneous components with respect to the word-length grading, we see that $ab_0 \in A$ and $ab_l \in \mathfrak{m}_A(X)$ for $l > 0$.

3. For $x, y \in X$, $\partial(xy) = \partial(x)y \pm x\partial(y)$. Writing $\partial(x) = \sum_{l \in \mathbb{N}} a_l$ as a sum of homogeneous components with respect to the word-length grading, since $A[X]$ is $\mathbb{D}$-local, $a_0 y \in \mathfrak{m}_A(X)$ and $a_l y \in (X^2)$ for $l > 0$. A symmetric argument gives that $x\partial(y) \in \mathfrak{m}_A(X) + (X^2)$.

For $A + K(X) + (X^2)$, we again check each term. For $a \in A$ or $ab \in K(X)$, the same arguments as above work (note that K is a DB ideal closed under the differential). For $x, y \in X$, the fact that the augmentation lifts to $A[X]$ and is a chain map implies that $\partial(A[X]_1) \subset K_0$. Writing $\partial(x) = \sum_{l \in \mathbb{N}} a_l$ as a sum of homogeneous components with respect to the word-length grading, this gives that $a_0 \in K$, and applying a symmetric argument to y gives that

$$\partial(xy) = \partial(x)y \pm x\partial(y) \subset K(X) + (X^2)$$

□

Example 3.2.15. The assumption that $A[X]$ is $\mathbb{D}$-local is essential. For example, let k be a field and $A = k[x \,|\, \partial(x) = 1]$. Let $B = A$, and let $\varphi\colon A \to B$ be the identity map. Consider the semifree extension $A[y, z \,|\, \partial(y) = \partial(z) = 1]$ with lift $\widetilde{\varphi}$ sending y and z to x. The kernel K in the above proposition is zero, and $\partial(yz) = z - y$. which is contained neither in $A + \mathfrak{m}_A X + (X^2)$ nor $A + KX + (X^2)$.

Definition 3.2.16. Let $A[X]$ be a $\mathbb{D}$-local semifree extension of DB R-algebras. The quotient complex

$$A[X]/(A + \mathfrak{m}_A(X) + (X^2)) = \cdots \to kX_n \to kX_{n-1} \to \cdots$$

is the *complex of indecomposables relative to* A, denoted $\operatorname{ind}_A A[X]$. We say that $A[X]$ is *minimal rel* A if $\operatorname{ind}_A A[X]$ is a minimal complex.

Now suppose that $\varphi\colon A \to S$ is an augmentation, and that the augmentation lifts to an augmentation $\widetilde{\varphi}\colon A[X] \to S$, and set $K = \ker(\widetilde{\varphi})$. The quotient complex

$$A[X]/(A + K(X) + (X^2)) = \cdots \to BX_n \to BX_{n-1} \to \ldots$$

is the *complex of indecomposables relative to* φ, denoted $\operatorname{ind}_\varphi A[X]$. We say that $A[X]$ is *minimal rel* φ if $\operatorname{ind}_\varphi A[X]$ is a minimal complex.

Definition 3.2.17. Let $A\langle X\rangle$ be a $\mathbb{D}$-local semifree Γ-extension of DB R-algebras. The quotient complex

$$A\langle X\rangle/(A + \mathfrak{m}_A(X^{(\geq 1)}) + (X^{(\geq 2)})) = \cdots \to kX_n \to kX_{n-1} \to \cdots$$

is the *complex of* Γ*-indecomposables relative to* A, denoted $\operatorname{ind}^\gamma_A A\langle X\rangle$. We say that $A\langle X\rangle$ is *minimal rel* A if $\operatorname{ind}^\gamma_A A\langle X\rangle$ is a minimal complex. Now suppose that $\varphi\colon A \to S$ is an augmentation, and that the augmentation lifts to an augmentation $\widetilde{\varphi}\colon A\langle X\rangle \to S$, and set $K = \ker(\widetilde{\varphi})$. The quotient complex $A\langle X\rangle/(A + K(X) + (X^{(\geq 2)}))$ is the *complex of indecomposables relative to* φ, denoted $\operatorname{ind}^\gamma_\varphi\langle X\rangle$. We say that $A\langle X\rangle$ is *minimal rel* φ if $\operatorname{ind}^\gamma_\varphi A\langle X\rangle$ is a minimal complex.

Remark 3.2.18. By the first isomorphism theorem, the complexes $\operatorname{ind}_\varphi A[X]$ and $\operatorname{ind}^\gamma_\varphi A\langle X\rangle$ are of free S-modules.

The additional structure of the $\mathbb{D}$-grading allows for minimality to be considered one internal degree at a time, as for chain complexes. Since DB R-algebras are concentrated in non-negative homological degrees and degrees coming from $\mathbb{D}$ (rather than $\mathbb{Z}$ and $G(\mathbb{D})$, as in definition 3.2.3), we consider minimality only in bidegrees from $\mathbb{N} \times \mathbb{D}$. In chapter 5 we will only need the notion of absolute minimality in particular degrees, and so will only define this notion.

Definition 3.2.19. Let A be a $\mathbb{D}$-local DB algebra, and let $i \in \mathbb{N}$, $d \in \mathbb{D}$ and $D \subset \mathbb{D}$. A is *absolutely minimal in bidegree* (i, d) if $\operatorname{ind} A$ is minimal in bidegree (i, d). A is *absolutely minimal in all D-degrees* if $\operatorname{ind} A$ is minimal in each bidegree contained in $\mathbb{N} \times D$, respectively.

The corresponding notion for Γ-extensions may be defined analogously.

Definition 3.2.20. Let A be a $\mathbb{D}$-local DB Γ-algebra, and let $i \in \mathbb{N}$, $d \in \mathbb{D}$ and $D \subset \mathbb{D}$. A is *absolutely Γ-minimal in bidegree* (i, d) if $\operatorname{ind}^\gamma A$ is minimal in bidegree (i, d). A is *absolutely Γ-minimal in all D-degrees* if $\operatorname{ind} A$ is minimal in each bidegree contained in $\mathbb{N} \times D$, respectively.

The following theorems can be found for the case of augmented DG algebras in [8, Lemma 7.2.2]. The proof in the $\mathbb{D}$-graded case requires only trivial modification (the key ingredient being Nakayama's lemma, which is applicable as all the algebras are $\mathbb{D}$-local).

Theorem 3.2.21. *$A[X]$ is a minimal model of $\varphi\colon A \to B$ if and only if*

1. *$X = X_{\geq 1, J}$, where $J \subset \mathbb{D}$ is the ideal $\{d + \deg(k) \mid k \in \ker(A \to B)\}$*
2. *$A[X]$ is minimal rel A,*
3. *$\widetilde{\varphi}\colon A[X] \to B$ is a quasi-isomorphism*

If $B = S$ is a ring concentrated in homological degree 0, then $A[X]$ is a minimal model of $\varphi\colon A \to S$ if and only if

1. *$X = X_{\geq 1,J}$, where $J \subset \mathbb{D}$ is the ideal $\{d + \deg(k) \mid k \in \ker(A \to S)\}$*
2. *$A[X]$ is minimal rel φ,*
3. *$\widetilde{\varphi}\colon A[X] \to S$ is a quasi-isomorphism*

The following is due to Gulliksen for the case of DG algebras. A complete proof may be found in [8, Chapter 6].

Theorem 3.2.22. *$A\langle X\rangle$ is an acyclic closure of $\varphi\colon A \to B$ if and only if*

1. *$X = X_{\geq 1,J}$, where $J \subset \mathbb{D}$ is the ideal $\{d + \deg(k) \mid k \in \ker(A \to B)\}$*
2. *$A\langle X\rangle$ is minimal rel A*
3. *$\widetilde{\varphi}\colon A\langle X\rangle \to B$ is a quasi-isomorphism*

If $B = S$ is a ring concentrated in homological degree 0, then $A\langle X\rangle$ is an acyclic closure of $\varphi\colon A \to S$ over A if and only if

1. *$X = X_{\geq 1,J}$, where $J \subset \mathbb{D}$ is the ideal $\{d + \deg(k) \mid k \in \ker(A \to S)\}$*
2. *$A\langle X\rangle$ is minimal rel φ*
3. *$\widetilde{\varphi}\colon A\langle X\rangle \to S$ is a quasi-isomorphism*

Acyclic closures are minimal in the sense of complexes in the following special case. It was originally proven by Gulliksen [24] and Schoeller [34]. The proof in [8, §6] works in the $\mathbb{D}$-graded case. The only necessary fact is that over the $\mathbb{D}$-field $R/\mathfrak{m}_R$, all modules are free (cf. [28, Proposition 4.1]), which is the key ingredient in [8, Proposition 6.2.7].

Theorem 3.2.23. *Let R be a $\mathbb{D}$-local ring with residue $R/\mathfrak{m}_R = k$. The acyclic closure $R\langle X\rangle$ of k over R is minimal as a complex, and hence is the minimal $\mathbb{D}$-graded free resolution of k over R.*

When extensions fail to be minimal, a standard procedure, due to Gulliksen for Γ-algebras, exists to produce a minimal semifree extension by quotienting out by a set of variables.

The following is [22, Lemma 3.2.1]

Lemma 3.2.24. *Let $R\langle Y\rangle$ be a $\mathbb{D}$-local semifree Γ-extension of a $\mathbb{D}$-graded ring. There exists a $\mathbb{D}$-local semifree Γ-extension $R\langle X\rangle$ and a surjective bigraded homomorphism $f\colon R\langle Y\rangle \to R\langle X\rangle$ satisfying*

1. *$R \hookrightarrow R\langle X\rangle$ is minimal rel R.*
2. *The map $\operatorname{ind}^{\gamma}_R R\langle Y\rangle \to \operatorname{ind}^{\gamma}_R R\langle X\rangle$ on indecomposable complexes induced by f is a quasi-isomorphism.*
3. *If the residue field of R has characteristic zero, then f is a quasi-isomorphism.*
4. *If the residue field of R has characteristic $p > 0$, then f induces $H_i(R\langle Y\rangle) \cong H_i(R\langle X\rangle)$ for $i < 2p$.*
5. *If the residue field of R has characteristic $p > 0$, and for all i $B_{2i}(R\langle Y\rangle) \subset R + \mathfrak{m}Y + Y^{(\geq 2)}$, then f is a quasi-isomorphism.*

Note that, in particular, $H_0(R\langle X\rangle) \cong H_0(R\langle Y\rangle)$ in all cases. Hence if $\varepsilon\colon R\langle Y\rangle \to S$ is an augmention, then ε induces an augmentation on $R\langle X\rangle$ turning it into an augmented algebra and the quotient $R\langle Y\rangle \to R\langle X\rangle$ into a map of augmented algebras.

Definition 3.2.25. The algebra $R\langle X\rangle$ in the above lemma is called the *minimization rel R* or simply the *minimization* of $R\langle Y\rangle$.

The following is [13, Proposition 1] in the case where $R = k$ is a field and $\mathbb{D} = \mathbb{N}^l$. The condition that $R[Y]$ be a $\mathbb{D}$-local extension then imposes that the internal degrees of the variables be non-zero. When allowing R to be an arbitrary ring, the same proof works.

Lemma 3.2.26. *Let $R[Y]$ be a $\mathbb{D}$-local semifree extension of a $\mathbb{D}$-graded ring. There exists a $\mathbb{D}$-local semifree extension $R[X]$ and a surjective bigraded homomorphism $f\colon R[Y] \to R[X]$ satisfying*

1. *$R[X]$ is minimal rel R*[1]*.*
2. *$H_0(R[X]) \cong H_0(R[Y])$.*
3. *The map $\operatorname{ind} R[Y] \to \operatorname{ind} R[X]$ on indecomposable complexes induced by f is a quasi-isomorphism.* [2]
4. *The induced map on homotopy Lie algebras is an isomorphism (See chapter 4)*
5. *Restricting the induced map f_* to the square-free multidegrees of $H(R[Y])$ yields a quasi-isomorphism*
6. *If the residue field of R has characteristic zero, then f is a quasi-isomorphism*

Note that, in particular, $H_0(R[X]) \cong H_0(R[Y])$ in all cases. Hence if $\varepsilon\colon R[Y] \to S$ is an augmention, then ε induces an augmentation on $R[X]$ turning it into an augmented algebra and the quotient $R[Y] \to R[X]$ into a map of augmented algebras.

Remark 3.2.27. Examination of the proofs reveals that the construction of $R[X]$ involves choosing bases of certain bigraded subspaces on $\operatorname{ind}^\gamma R\langle Y\rangle$ and $\operatorname{ind} R[Y]$. [22,

[1]Since in [13, Proposition 1] R_0 is presumed to be a field and since, in general, minimality rel k implies absolute minimality, this part is simply stated as absolute minimality therein

[2]this is not included in [13, Proposition 1], but we place it here to be more consistent with [22, Lemma 3.2.1].

Lemma 1.8.7] and [12, Lemma 1] give that different choices induce automorphisms of $R\langle Y\rangle$ and $R[Y]$, respectively, which in turn induce isomorphisms on the quotients. Hence, while not explicitly stated in the results in these theorems, the algebras $R\langle X\rangle$ and $R[X]$ are unique up to isomorphism (of augmented algebras, if appropriate). This justifies the following definition:

Definition 3.2.28. The algebra $R[X]$ in the above lemma is called the *minimization rel* R of $R[X]$.

Chapter 4

The Bigraded Deviations and Homotopy Lie Algebra

As with Betti numbers, minimal models and acyclic closures allow for numerical invariants to be associated to rings and ring homomorphisms. For these to be invariants, it is necessary that minimal models and acyclic closures be unique up to isomorphism. The following are proven in [22, p. 1.9.5] (for acyclic closures) and [8, p. 7.2.3] (for minimal models). The key ingredient is Nakayama's lemma, which still works in our setting despite the fact that $\mathbb{D}$-fields need not be fields in the ordinary sense2.1.8, so the same proofs work.

Theorem 4.0.1. *1. Let $\varphi\colon (A, I) \to (B, B_{\geq 1})$ be a surjective $\mathbb{D}$-local bigraded map of $\mathbb{D}$-local DB Γ R-algebras, in which $H_i(A)$ and $H_i(B)$ are finitely generated for each i. Then any two acyclic closures $A\langle X\rangle$ and $A\langle Y\rangle$ of φ are isomorphic. The isomorphism induces an isomorphism of complexes $\operatorname{ind}^{\gamma}_{A} A\langle X\rangle \to \operatorname{ind}^{\gamma}_{A} A\langle Y\rangle$ and hence $\#X_{i,j} = \#Y_{i,j}$ for each $(i, j) \in \mathbb{N} \times \mathbb{D}$.*

2. Isomorphisms of rings lift to isomorphisms of acyclic closures, in the following sense: If the following diagram commutes:

$$\begin{array}{ccc} R & \xrightarrow{\cong} & R' \\ \downarrow & & \downarrow \\ S & \xrightarrow{=} & S \end{array}$$

then the acyclic closure $R\langle X\rangle$ *of* S *over* R *is isomorphic to the minimal model* $R'\langle Y\rangle$ *of* S *over* R'*, and the isomorphism induces an isomorphism of complexes* $\operatorname{ind}^{\gamma} R\langle X\rangle \to \operatorname{ind}^{\gamma} R'\langle Y\rangle$.

Theorem 4.0.2. 1. *Let* $\varphi\colon A \to B$ *be a surjective bigraded map of* $\mathbb{D}$*-local DB* R*-algebras, in which* $H_i(A)$ *and* $H_i(B)$ *are finitely generated for each* i*. Then any two minimal models* $A[X]$ *and* $A[Y]$ *are isomorphic. The isomorphism induces an isomorphism of complexes* $\operatorname{ind}_A A[X] \to \operatorname{ind}_A A[Y]$ *and hence* $\#X_{i,j} = \#Y_{i,j}$ *for each* $(i,j) \in \mathbb{N} \times \mathbb{D}$.

2. *Isomorphism of rings lift to isomorphisms of minimal models, in the following sense: If the following diagram commutes:*

$$\begin{array}{ccc} R & \xrightarrow{\cong} & R' \\ \downarrow & & \downarrow \\ S & \xrightarrow{=} & S \end{array}$$

then the minimal model $R[X]$ *of* S *over* R *is isomorphic to the minimal model* $R'[Y]$ *of* S *over* R'*, and the isomorphism induces an isomorphism of complexes* $\operatorname{ind} R[X] \to \operatorname{ind} R'[Y]$.

Uniqueness ensures that the following numerical invariants are well-defined.

Definition 4.0.3. Let R be a $\mathbb{D}$-local ring and let $(i,j) \in \mathbb{N} \times \mathbb{D}$. The *deviation of* R *in bidegree* (i,j), denoted $\varepsilon_{i,j}(R)$, is the number of elements of X with bidegree (i,j) in an acyclic closure $R\langle X\rangle$ of $R/\mathfrak{m}_R$ over R.

Definition 4.0.4. Let $\varphi\colon R \to S$ be a surjective map of $\mathbb{D}$-local rings and let $(i,j) \in \mathbb{N}_{>1} \times \mathbb{D}$. The *deviation of* φ *in bidegree* (i,j), denoted $\varepsilon_{i,j}(\varphi)$, is the number of elements of X in bidegree $(i-1,j)$ in a minimal model $R[X]$ of φ.

The reason for the seemingly peculiar choices of acyclic closures and minimal models in the above definitions is due to the following computational result comparing the two, due to Avramov. The proof in [8, p. 7.2.6] holds with only trivial modification needed to account for internal degrees.

Theorem 4.0.5 (Avramov). *Let R be a $\mathbb{D}$-local ring with residue $k = R/\mathfrak{m}_R$, and suppose $R' = \widehat{R_0} \otimes_{R_0} R$ is presentable. Let $\varphi\colon Q \to R'$ be a minimal presentation of R as a quotient of a $\mathbb{D}$-local polynomial ring $Q = Q_0[x_1, \dots, x_l]$. Let $R\langle X\rangle$ be an acyclic closure of k and $Q[Y]$ be a minimal model of φ. Then for each $i \geq 1$, there is a quasi-isomorphism of $\mathbb{D}$-graded R algebras*

$$R\langle X_{\leq i}\rangle \simeq k[Y_{\geq i}] \cong Q[Y]/((Y_{<i}) + \mathfrak{m}_Q).$$

Hence

1. $\varepsilon_{1,0}(R) = \dim Q_0 = \operatorname{edim} R_0$.
2. *When $j \neq 0$, $\varepsilon_{1,j}(R)$ is the number of x_t's in the list $x_1, \dots x_l$ which are of degree j.*
3. *When $i > 1$, $\varepsilon_{i,j}(R) = \varepsilon_{i,j}(\varphi)$.*

Remark 4.0.6. The description of $\varepsilon_{1,j}(R)$ comes from the fact that a set of minimal generators of $\mathfrak{m}_Q$ are in degree-preserving bijective correspondence with a set of minimal generators of $\mathfrak{m}_R$. The connection between the quasi-isomorphisms $R\langle X_{\leq i}\rangle \simeq k[Y_{\geq i}]$ and the equalities $\varepsilon_{i,j}(R) = \varepsilon_{i,j}(\varphi)$ for $i > 1$ is due to the homology of absolutely minimal semifree extensions of fields. In particular, when $k[Y_{\geq i}]$ is absolutely minimal then $H_i(k[Y_{\geq i}]) \cong kY_i$ [8, Rem. 7.2.1], and so $H_i(R\langle X_i\rangle) \cong kY_i$. Hence the adjunction of X_{i+1} in the next step of the acyclic closure induces a bijection $X_{i+1} \leftrightarrow Y_i$ that preserves $\mathbb{D}$-degrees. This phenomenon is responsible for the indexing shift between definition 4.0.3 and definition 4.0.4.

theorem 4.0.5 can be used to relate the deviations of a ring to classification of its singularity type. The second part of the following statement is a classical theorem by Assmus in the local case [2, Theorem 2.7].

Corollary 4.0.7. *R is a regular ring if and only if $\varepsilon_{ij}(R) = 0$ for all $i \geq 2$, and R is a complete intersection ring if and only if $\varepsilon_{ij}(R) = 0$ for all $i \geq 3$.*

Proof. Regularity and the complete intersection property are invariant under completion, so we may assume R_0 is complete. Let $Q[X]$ be a minimal model of R. Applying the theorem, $\varepsilon_{ij}(R) = 0$ for $i \geq 2$ implies that $X = \emptyset$, set $R \cong Q$ is a regular ring. $\varepsilon_{ij}(R) = 0$ for $i \geq 3$ implies that $X = X_1$, so $Q[X]$ is the Koszul complex on a minimal generating set of $\ker(Q \to R)$. $Q[X]$ is a resolution of R over a regular ring, and so the minimal generating set of $\ker(Q \to R)$ is a regular sequence. Hence R is a complete intersection. □

4.1 Bigraded Lie Algebras

In this section, suppose that $\mathbb{D}$ has no non-trivial units, and let k be a field. Let R be a $\mathbb{D}$-graded ring with $R_0 = k$. Then, when $Q[X]$ is a minimal model of R as in theorem 4.0.5, the sequence of $\mathbb{D}$-graded k-spaces $\{kX_i\}_{i\in\mathbb{N}}$ uniquely determines a bigraded Lie algebra. Conversely, bigraded Lie algebras determine minimal extensions of k. The goal of this section is to discuss this correspondence in the $\mathbb{D}$-graded case. Analogously to section 2.2, we first define bigraded Lie algebras and establish some notational conventions. The classical case (without $\mathbb{D}$-grading) may be found in [3] and [8, §10]. The $\mathbb{N}^l$-graded case is described in [12]. We will follow the approach found in [16] which emphasizes connections to Koszul duality.

Definition 4.1.1. Let k be a field A *homologically* $\mathbb{D}$*-bigraded* (or just "bigraded") Lie algebra over k is an $\mathbb{N} \times \mathbb{D}$-graded k module L equipped with two operations:

1. A k-bilinear form $[\,,\,]\colon L \otimes_k L \to L$, called the *bracket* of L, which is *bigraded*: $[L_{i,j}, L_{n,m}] \subset L_{i+n,j+m}]$ for all $(i,j),(n,m) \in \mathbb{N} \times \mathbb{D}$

2. A family of k-linear maps $L_{(2i+1),j} \to L_{4i+2,2j}$ called the *reduced square* and denoted $x \mapsto x^{[2]}$

which satisfy the following axioms for all bihomogeneous elements $x, y, z \in L$:

(1) (degree-wise finite generation) Each L_i is finitely generated (and hence free) over k.

(2) (positivity) $L_{0,*} = L_{*,0} = 0$

(3) (graded-anticommutativity) $[x,y] = -(-1)^{|x||y|}[y,x]$

(4) (compatibility) Whenever $|x|$ and $|y|$ are odd, $(x+y)^{[2]} = x^{[2]} + y^{[2]} + [x,y]$

(5) the *Jacobi identities*:

a) (Jacobi identity) $[x,[y,z]] = [[x,y],z] + (-1)^{|x||y|}[y,[x,y]]$

b) When $|x|$ is odd and $\lambda \in k$, $(\lambda x)^{[2]} = \lambda^2 x^{[2]}$

c) When $|x|$ is odd, $[x^{[2]}, y] = [x,[x,y]]$

d) When $|x|$ is odd, $[x,[x,x]] = 0$

e) When $|y|$ is even, $[y,y] = 0$

Remark 4.1.2. Only the first of the Jacobi identities is likely to be recognizable by the reader familiar with the classical definition of ungraded Lie algebras. The rest are only necessary in certain characteristics. For example, the conditions $[x,x] = 0$

and $[y,[y,y]]=0$ in the above are superflous when 6 is invertible in k. Similarly, the inclusion of the quadratic square is only necessary in characteristic two. Compatibility and the condition $(\lambda x)^{[2]}=\lambda^2 x^{[2]}$ give that $2x^{[2]}=[x,x]$ for all x of odd homological degree, and so

- The quadratic square is completely determined by the bracket when $\mathrm{char}(k)\neq 2$.
- $[x,x]=0$ when $\mathrm{char}(k)=2$.

The analogues of the ordinary structures from algebras are available.

Definition 4.1.3.

1. A *morphism* of Lie algebras is a map of bigraded k-modules which commutes with the bracket and reduced square operations.
2. A *subalgebra* of a Lie algebra L is a subset closed under the bracket and reduced square operations.
3. A bigraded submodule $M\subset L$ is an *ideal* of L if $[M,L]\subset M$ and M is closed under reduced squares. The quotient k module is then a bigraded Lie algebra. Note that ideals are also subalgebras, since $[M,M]\subset M$.
4. The subalgebras (and ideals) generated by a set $S\subset L$ are the intersection of all subalgebras (ideals) containing the set S.

4.2 Correspondence with Quadratic Semifree Extensions

In this section, suppose $\mathbb{D}$ has no non-trivial units. In this setting, $\mathbb{D}$-bigraded Lie algebras are obtainable from absolutely minimal semifree extensions of k. In fact, there is an equivalence of categories between bigraded Lie algebras and minimal semifree extensions of k satisfying additional properties, which we now define.

Definition 4.2.1. Let $k[X]$ be a $\mathbb{D}$-local semifree extension of a field k. We say that $k[X]$ is *quadratic* if $\partial(X) \subset kX^2$, where kX^2 is the k-space spanned by words of length exactly two, as defined in 3.2.11.

Remark 4.2.2. Note that in the above definition, X_0 may be non-empty. The stipulation that $k[X]$ be $\mathbb{D}$-local then ensures that $X_{0,0} = 0$. In establishing the correspondence with bigraded Lie algebras, this is essential to ensuring positivity of the resulting Lie algebra. It is also important to emphasize in the above definition that kX^2 is the space spanned by words of length two, rather than the ideal (X^2). Hence $k[X]$ quadratic implies that $k[X]$ is minimal rel k, but not conversely. This is further clarified in definition 4.3.1.

Definition 4.2.3. *Trigraded morphisms* $f\colon k[X] \to k[Y]$ of quadratic semifree extensions are bigraded morphisms of DB algebras which also respect the word-length grading. Quadratic extensions and trigraded morphisms form the *category of quadratic semifree extensions of k*

The following theorem is classical for DG algebras, due to Quillen [32] in characteristic zero and Avramov [3] in positive characteristic.

Theorem 4.2.4. *There is an equivalence of categories between quadratic semifree extensions and bigraded Lie algebras.*

We will say a few remarks about the proof to show why this result still holds in the bigraded setting. For convenience in dealing with the degree shifting involved, we will introduce the following notation.

Definition 4.2.5. Let k be a field and X be an $\mathbb{N} \times \mathbb{D}$-bigraded set. The *tensor algebra* on X, denoted $T(X)$ is the the free algebra on X (with no assumption of graded commutivity). It has a direct sum decomposition

$$T(X) = k \oplus kX \oplus (kX)^{\otimes 2} \oplus \dots$$

and k-bilinear multiplication defined on pure tensors by

$$(x_1 \otimes \cdots \otimes x_n) \cdot (y_1 \otimes \cdots \otimes y_m) = x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_m.$$

It is additionally graded by word length; the k-span of the set of words of length n is denoted by $T^n(X)$.

Definition 4.2.6. The *free (strictly) graded-anticommutative k-algebra on X*, denoted $\mathcal{A}(X)$, is the exterior algebra on X_{2i} tensored over k with the polynomial algebra on X_{2i+1}. In other words, it is the quotient of $T(X)$ by the ideal generated by the graded anti-commutators $xy - (-1)^{|x||y|}yx$ and by the elements z^2 where z ranges over all elements of X of even homological degree. It is additionally graded by word length; the k-span of the set of words of length n is denoted by $\mathcal{A}^n(X)$.

The relation between the words of length n in $k[X]$ and $\mathcal{A}[X]$ is encoded in the shift: $\mathcal{A}^n(\Sigma X) \cong \Sigma^n(kX^n)$.

The equivalence proceeds as follows: starting with a quadratic semifree extension $k[X]$, set $\pi_*(k[X]) = \Sigma(kX)$. Note that this is a bigraded k-space concentrated in positive homological degrees. Since $k[X]$ is quadratic, the Leibniz rule implies $\partial(kX^l) \subset kX^{l+1}$, and in particular, $\partial(kX^2) \subset kX^3$, which gives the top row in the following diagram defined for each homological degree i; the map Δ in bottom row of the diagram is obtained by imposing commutativity of the lefthand square:

$$\begin{array}{ccccc} kX_{i-1} & \xrightarrow{\partial} & kX^2_{i-2} & \xrightarrow{\partial} & kX^3_{i-3} \\ \downarrow{\cong} & & \downarrow{\cong} & & \downarrow{\cong} \\ (\pi_i(k[X])) & \xrightarrow{\Delta} & \mathcal{A}^2(\pi_i(k[X])) & \xrightarrow{\Delta\wedge 1 - 1\wedge\Delta} & \mathcal{A}^3(\pi_i(k[X])) \end{array} \qquad (*)$$

Using the Leibniz rule to calculate $\partial(kX^2)$ shows that the righthand square in the diagram also commutes. Commutativity of the whole diagram and the condition $\partial^2 = 0$ then gives that $(\Delta \wedge 1 - 1 \wedge \Delta)\Delta = 0$.

Let $\pi^*(k[X])$ be the homologically graded dual of $\pi_*(k[X])$ (i.e., $\pi^*(k[X]) = \oplus_{i\geq 0} \operatorname{Hom}_k(\pi_i(k[X]), k)$) The dual of the projection $T(\pi_*(k[X])) \to \mathcal{A}(\pi_*(k[X]))$ is an embedding of $\mathcal{A}(\pi_*(k[X]))^*$ into $T(\pi_*(k[X]))^*$.

Similarly to the more classical symmetrization map as described in [19, A2.4], the antisymmetrization map $\mathsf{a}\colon \pi^*(k[X]) \otimes \pi^*(k[X]) \to T(\pi_*(k[X]))^*$ defined by

$$\mathsf{a}(x \otimes y) = x \otimes y - (-1)^{|x||y|} y \otimes x$$

, maps surjectively onto the image of $\mathcal{A}(\pi_*(k[X]))^*$ whenever $\operatorname{char}(k) \neq 2$. In this case, postcomposing a with the dual of Δ defines the bracket of $\pi^*(k[X])$, which in turn determines the reduced square as described in remark 4.1.2. When $\operatorname{char}(k) \neq 2$, $\Delta^*\mathsf{a}$ defines the bracket, but a fails to be surjective: its image in $T(\pi_*(k[X]))^*_2$ fails to include the elements $x \otimes x$ (but no others). The reduced square is defined by composing the diagonal map $d\colon \pi^*_{\text{odd}}(k[X]) \to T(\pi_*(k[X]))^*$, defined by $d(x) = x \otimes x$, with the dual of Δ.

In either case, we have defined a bracket and reduced square map on $\pi^*(k[X])$. As the maps Δ^*, d, and a are all bigraded, the bracket and reduced square on $\pi^*(k[X])$ are bigraded. The Jacobi identities of a bigraded Lie algebra may be verified using the equation $(\Delta \wedge 1 - 1 \wedge \Delta)\Delta = 0$. Antigraded commutivity and compatibility hold for the antisymmetricization and diagonal maps a and d.

Going the other direction, the above steps are all reversible: given a bigraded Lie algebra L, set $X = \Sigma^{-1}L^*$. The semifree extension $k[X]$ is defined by using Δ and imposing commutativity to define the differential in the top row of the diagram $(*)$. Positivity gives that $X_{0,0} = 0$ and since $\mathbb{D}$ has no non-trivial units, this suffices for $k[X]$ to be $\mathbb{D}$-local.

Furthermore, the steps outlined above are all functorial: trigraded morphisms of quadratic extensions induce a map of bigraded Lie algeras. When starting with a map $f\colon L \to K$ of bigraded Lie algebras, following the steps outlined above in reverse induces a map on k-spaces $\Sigma^{-1}(K^*) \to \Sigma^{-1}(L^*)$ which extends uniquely to a trigraded morphism of quadratic extensions. That it is a chain map follows from the fact that f commutes with the bracket of L, and that the differential is defined using this bracket.

Remark 4.2.7. The k-space $\pi_*(k[X])$ together with Δ in the above construction is an example of a *bigraded Lie coalgebra*. We will not make explicit use of this structure.

4.3 Homotopy Lie Algebras of Rings and Ring Homomorphisms

In this section, $\mathbb{D}$ will have no non-trivial units. When $A = Q[X]$ is a $\mathbb{D}$-local semifree extension of a ring Q, each of the ideals $\mathfrak{m}_A^l$ are bigraded, and so each $\mathfrak{m}_A^l/\mathfrak{m}_A^{l+1}$ is a bigraded A-module. When A is absolutely minimal, the Leibniz rule gives that $\partial(\mathfrak{m}_A^l) \subset \mathfrak{m}_A^{l+1}$. This allows us to make the following definition:

Definition 4.3.1. Let $A = Q[X]$ be a $\mathbb{D}$-local semifree extension of Q. The *associated trigraded ring of* $A^\natural$, denoted $\operatorname{gr} A^\natural = \oplus_{l\in\mathbb{N}}\mathfrak{m}_A^l/\mathfrak{m}_A^{l+1}$. The homological and $\mathbb{D}$-gradings are inherited from those of A. The third grading is provided by the index l.

The differential may be defined on the trihomogeneous elements of $\operatorname{gr} A^\natural$ by $\partial(a + \mathfrak{m}_A^l) = \partial(a) + \mathfrak{m}_A^{l+1}$. It is well defined since $\partial(\mathfrak{m}_A^l) \subset \mathfrak{m}_A^{l+1}$. It satisfies the Leibniz rule since ∂ does on A.

If Q is furthermore assumed to be a regular ring, $\operatorname{gr} Q[X]_{0,*,*} \cong k[x_1, \ldots, x_d]$, where $d = \dim Q$, so $\operatorname{gr} A_{0,0,0} = k$. In higher homological degrees, the image of X in degrees $(*, *, 1)$ determines a set of variables over $k[x_1, \ldots, x_d]$, which allows $\operatorname{gr} A$ with the differential above to be identified with a semifree extension of $k[x_1, \ldots, x_d]$. Combining the sets of variables in all homological degrees, $\operatorname{gr} A$ equipped with the differential described above may be identified with a semifree extension of k called the *associated trigraded differential algebra of A.*

Furthermore, $\partial(X) \subset X^2$ in $\operatorname{gr} A$ since all higher word-length terms land in the denominators of the quotients. Hence $\operatorname{gr} A$ is a quadratic extension. Whenever $f \colon A \to B$ is a bigraded $\mathbb{D}$-local homomorphism, $f(\mathfrak{m}_A) \subset \mathfrak{m}_B$ ensures that the map $\operatorname{gr} f \colon \operatorname{gr} A \to \operatorname{gr} B$ defined by $\operatorname{gr} f(a + \mathfrak{m}_A^l) = f(a) + \mathfrak{m}_B^l$ is well-defined trigraded map of algebras. It commutes with the differential since f does. Altogether, we have proven the following:

Proposition 4.3.2. *Let Q be a $\mathbb{D}$-local regular ring with $Q/\mathfrak{m}_Q = k$. $Q[X] \mapsto \operatorname{gr} Q[X]$ determines a functor from $\mathbb{D}$-local semifree extensions of Q to $\mathbb{D}$-local quadratic extensions of k.*

This allows us to use the correspondence between quadratic extensions and Lie algebras to attach a well-defined Lie algebra to arbitrary minimal models of rings and ring homomorphisms.

Definition 4.3.3. Let $Q[X]$ be an absolutely minimal semifree extension with Q regular. Then $\pi^*(Q[X])$ is defined to be the Lie algebra $\pi^*(\operatorname{gr} Q[X])$ as defined in theorem 4.2.4.

In a factorization $R \to R' \to S$, the ring R' is not necessarily regular so we can not apply this idea to the minimal model $R'[X]$ directly. This is solved by quotienting by the maximal ideal of R.

Definition 4.3.4. Let $\varphi\colon R \to S$ be a $\mathbb{D}$-local bigraded homomorphism with $R_0 = k$ and $S_0 = l$ both fields, and let $R \to R' \xrightarrow{\tilde{\varphi}} S$ be a minimal standard factorization. Let $R'[X]$ be a minimal model of $\tilde{\varphi}$. Then $F^\varphi = k \otimes_R R'[X]$ is a minimal semifree extension of k, since $F_0^\varphi = k \otimes_R R'$ is a polynomial ring over k. Furthermore, since $R'[X]$ is a resolution of S over R, F^φ is a DB algebra model for the left-derived tensor product $k \otimes_R^L S$ (so $H(F^\varphi) = \operatorname{Tor}^R(S, k)$). The algebra F^φ is called the *homotopy fiber of* φ.

Definition 4.3.5.

1. Let $\varphi\colon R \to$ be $\mathbb{D}$-local bigraded homomorphism with R_0 and S_0 both fields, and let F^φ be the homotopy fiber of φ. The *homotopy Lie Algebra of* φ, denoted $\pi^*(F^\varphi)$, is the bigraded Lie algebra $\pi^*(\operatorname{gr}(k \otimes_R R'[X]))$ as defined in theorem 4.2.4.

2. Suppose $R_0 = k$ and R is finitely generated over k. Let $\varphi\colon k \to R$ be the inclusion, and F^φ be the homotopy fiber. Then $\pi^*(R) = \pi^*(F^\varphi)$ is the *homotopy Lie Algebra of* R.

In the case of local rings, uniqueness of the homotopy Lie algebra follows from more powerful results proven by Avramov [3, Theorem 4.2] for $\pi^{\geq 2}(F^\varphi)$ and extended to $\pi^1(F^\varphi)$ by Briggs [16, Theorem 15]. In contrast, our approach is to use the uniqueness results already proven in this work for the technical constructions involved. It is likely that an approach along the same lines as those employed in the local case would work, but the details remain to be checked.

Theorem 4.3.6. *Suppose that $\mathbb{D}$ has no non-trivial units, and let $R \to S$ be a $\mathbb{D}$-local homomorphism with $R_0 = k$ and $S_0 = l$ both fields. Then $\pi^*(F^\varphi)$ is unique up to isomorphism of bigraded Lie algebras.*

Proof. Let $R \to T \to S$ and $R \to U \to S$ be two minimal standard factorizations. By proposition 2.1.18, there is a $\mathbb{D}$-local isomorphism $\psi\colon T \to U$ of augmented l-algebras. It lifts to an isomorphism of minimal models by theorem 4.0.2, and tensoring down yields an isomorphism $\widehat{\psi}\colon l[X] \to l[Y]$.

Then $\mathrm{gr}(\widehat{\psi})\colon \mathrm{gr}(l[X]) \to \mathrm{gr}(l[Y])$ is an isomorphism of quadratic semifree extensions. Applying π^* yields an isomorphism of homotopy Lie algebras. □

Applying this result to the inclusion $k \to R$, we get

Corollary 4.3.7. $\pi^*(R)$ *is unique up to an isomorphism of bigraded Lie algebras.*

When φ is a complete intersection homomorphism, or when R is a complete intersection, $\pi^*(F^\varphi)$ and $\pi^*(R)$ can be computed explicitly. For simplicity, we describe a special case.

Example 4.3.8. Suppose the factorization of φ has the form $R \to R[x_1, \ldots, x_n] \to R[x_1, \ldots, x_n]/(x_1^l, \ldots, x_n^l)$ for some integer $c > 1$, with $\deg(x_i) = d_i$. The minimal model for $R[x_1, \ldots, x_n] \to R[x_1, \ldots, x_n]/(x_1^j, \ldots, x_n^j)$ is the Koszul complex

$$R[x_1, \ldots, x_n, y_1, \ldots, y_n \,|\, \partial(y_i) = x_i^j]$$

and the homotopy fiber is $k[x_1, \ldots, x_n, y_1, \ldots, y_n]$. Hence $\pi^1(F^\varphi)_{d_i} = k(\Sigma x_i^*)$ and $\pi^2(F^\varphi)_{d_i} = k(\Sigma y_i^*)$.

When $l > 3$, the quadratic part of the differential is zero, so $\pi^*(F^\varphi)$ has trivial bracket and reduced square operations. When $l = 2$, the homotopy fiber is quadratic, and the map Δ applied to $(\Sigma x_i^*)^2$ is Σy_i^*, so the reduced square operation is defined by $q(\Sigma x_i^*) = \Sigma y_i^*$ and the bracket is trivial.

More generally, when ϕ is surjective, $Y_0 = \emptyset$, the first step $A[Y_1]$ of the above construction is the Koszul complex on a minimal generating set of $\ker(\phi)$, and the second step is the adjunction of a set of variables in bijection with a minimal set of generators of $H_1(A[Y_1])$. This observation will be key in section 6.2. Already, this allows us to extend corollary 4.0.7 to the relative situation, giving us a criterion for the complete intersection property of a homomorphism in terms of the deviations.

Proposition 4.3.9. *Let $\varphi\colon R \to S$ be a $\mathbb{D}$-local homomorphism with $R_0 = k$ and $S_0 = l$ both fields. Then $\varepsilon_{ij}(\varphi) = 0$ for $i \geq 3$ if and only if φ is a complete intersection homomorphism.*

Combining all of the results so far with theorem 3.2.23 and theorem 4.0.5, we already have

Proposition 4.3.10. *Let R be a $\mathbb{D}$-graded ring with $R_0 = k$ a field. Then $\varepsilon_{ij}(R) = \dim_k \pi^i(R)_j$. Let $\varphi\colon R \to S$ be a $\mathbb{D}$-local homomorphism with $S_0 = l$. Then $\varepsilon_{ij}(\varphi) = \dim_l \pi^i(F^{\varphi})_j$ for each $(i,j) \in \mathbb{N} \times \mathbb{D}$.*

In fact, this correspondence can be strengthened: $\mathrm{Ext}_R(k,k)$ has the structure of a (typically non-commutative) bigraded k-algebra under composition product. The graded commutator defines a bigraded Lie algebra structure on $\mathrm{Ext}^{\geq 1}(k,k)$. The adjoint to this forgetful functor is the *universal enveloping algebra functor*, which takes a Lie algebra L and builds an (ordinary) algebra UL so that the commutator of UL coincides with the bracket of L. We end with the following important theorem which relates the homotopy Lie algebra to the Ext algebra. The complete proof requires many additional ingredients beyond those discussed in this thesis. A general proof of the ungraded case can be found in [16, §2.2, §3.1]. This proof appears to generalize to the case when $\varphi\colon R \to S$ is a $\mathbb{D}$-local morphism with $\mathbb{D}$ having no non-trivial units and R_0, S_0 fields, by adding $\mathbb{D}$-degrees in the arguments. The details remain to be checked, but we will not make explicit use of this result so we leave this to future work. We include the result here to further explain the significance of the homotopy Lie algebra.

Theorem 4.3.11. *Let $\varphi\colon R \to S$ be a local homomorphism and let F^φ be the homotopy fiber. Then $U\pi^*(F^\varphi) \cong \operatorname{Ext}_{F^\varphi}(l, l)$. In particular, when $\varphi\colon Q \to \widehat{R}$ is a minimal Cohen presentation of R, $F^\varphi \simeq R$, so $U\pi^*(R) \cong \operatorname{Ext}_R(k, k)$.*

4.4 Long Exact Sequences of Homotopy Lie Algebras

In this section, $\mathbb{D}$ will have no non-trivial units, $\varphi\colon R \to S$ will be a $\mathbb{D}$-local homomorphism, and we will assume that all factoriziations are standard. Sometimes, the homotopy Lie algebras of R, S, and φ organize into an exact sequence, allowing information about them to be related.

Definition 4.4.1. Let $\varphi\colon R \to S$ be a $\mathbb{D}$-local homomorphism with $R_0 = k$ and $S_0 = l$ both fields. We say φ *induces an exact sequence in homotopy Lie algebras* if an exact sequence of $\mathbb{D}$-graded k-vector spaces of the form

$$\cdots \leftarrow \pi^i(R) \otimes_k l \leftarrow \pi^i(S) \leftarrow \pi^i(F^\varphi) \leftarrow \dots$$

If such an exact sequence exists of k-vector spaces in each fixed internal degree $j \in D$ for some subset D, then φ *induces exact sequences in homotopy Lie algebras in all D-degrees.*

In this situation the structure of $\pi^* F^\phi$ may be used to study the transfer of properties of R along ϕ to properties of S. The mechanism underlying this comparison is rather technical, and has only been discussed in the local case, so we outline the ideas below.

The strategy behind creating such an exact sequence is to begin with a minimal presentation $k[X_0] \to R$ of R and form the minimal factorization $k[X_0] \to S$ as $k[X_0] \to l[X_0, Y_0] \to S$. Form the minimal model $k[X]$ of R over $k[X_0]$, and extend coefficients up to $l[X_0, Y_0]$ to get a semifree extension $l[X, Y_0]$ with a surjection $l[X, Y_0] \twoheadrightarrow S$. Let $l[X, Y]$ be the minimal model of S over $l[X, Y_0]$, and let $l[W]$ be the minimal model of S. $l[X, Y]$ can be compared with the homotopy fiber of φ: $k \otimes_{k[X]} l[X, Y] \cong l[Y]$ is absolutely minimal and $k \otimes_{k[X]} l[X, Y] \simeq k \otimes_R S$ is the homotopy fiber of φ. Hence, if $l[X, Y] \cong l[W]$, then we would have $X_{i,j} + Y_{i,j} = W_{i,j}$ for each bidegree i, j, a much stronger condition then the exact sequence in definition 4.4.1. However, such an isomorphism fails in general: $l[X, Y]$ is only minimal rel $l[X, Y_0]$, and need not be absolutely minimal.

The following diagram summarizes the objects described above.

$$\begin{array}{ccccc} k[X] & \longrightarrow & l[X,Y] & \longrightarrow & l[Y] \\ \downarrow{\simeq} & & \downarrow{\simeq} & & \downarrow{\simeq} \\ R & \longrightarrow & S & \longrightarrow & S/\mathfrak{m}_R S \end{array}$$

Taking indecomposables and extending to l-coefficients yields an exact sequence of complexes of k-spaces:

$$0 \to \operatorname{ind}_{k[X]} \to \operatorname{ind}_{l[X,Y]} \to \operatorname{ind}_{l[Y]} \to 0$$

So if there is a quasi-isomorphism $f \colon l[X, Y] \to l[W]$ so that the induced map $(\operatorname{ind} f)_{*,j}$ is a quasi-isomorphism, then $H_i(\operatorname{ind} l[W])_j$ can replace $H_i(\operatorname{ind} l[X, Y])_j$ for each i in the long exact sequence in homology arising from the exact sequence above. Shifting and dualizing then gives an exact sequence in homotopy Lie algebras in internal degree j.

In particular, when $\operatorname{char} k = \operatorname{char} l = 0$, the minimization of $l[X, Y]$ rel l is absolutely minimal and acyclic by lemma 3.2.26 and hence is isomorphic to $l[W]$. Hence we obtain the following proposition, which is classical[22, Theorem 3.2.4] in the local case.

Proposition 4.4.2. *Let $\varphi\colon R \to S$ be a $\mathbb{D}$-local homomorphism and suppose $R_0 = k$ is a field of characteristic zero. Then φ induces an exact sequence in homotopy Lie algebras.*

The map φ need not always induce exact sequences in homotopy Lie algebras, as evidenced by the following example:

Example 4.4.3. Let $\operatorname{char} k = p > 0$, and let $R = k[a]/(a^4)$, $S = k[a]/(a^2)$, and $\phi : R \to S$ be the projection. Since R and S are both complete intersections $\pi^i(R) = \pi^i(S) = 0$ for $i \geq 3$. A minimal model for S over R begins with $R[x_1, x_2 \,|\partial x_1 = a^2, \partial x_2 = a^2 x_1]$. However, by the Leibniz rule $\partial(x_2^p) = pa^2 x_1 f^{p-1} = 0$, so both x_2^p and $\partial(x_2)x_2^{p-1}$ are non-trivial cycles of degree $2p$ and $2p-1$, respectively. Hence the construction of the minimal model continues with the adjunction of variables in degree $2p+1$ and $2p$, and so $\varepsilon_{2p+2}(\phi) \neq 0$ (in fact, by iterating this argument, the minimal model will involve the adjunction of an infinite number of variables). As $2p+2 > 3$, the existence of the long exact sequence in homotopy Lie algebras would result in a contradiction since $0 \to \pi^{2p+2}(F^\phi) \to 0$ would be exact.

Remark 4.4.4. In the classical local case, long exact sequences in homotopy Lie algebras are known to be induced by Golod morphisms, when S has a finite resolution by flat R-modules, and a variety of other cases [16, § 3.5]. These results are likely to hold in the graded case as well, but the details remain to be checked.

Chapter 5

Calculating Deviations with Partial Models and Partial Closures

In his thesis [12] and followup paper [13], Berglund shows that when the relations of R are generated by monomials, the bigraded deviations may be computed by examining the homology of some simplicial complexes associated to a minimal set of monomial generators. In order to prove these results, Berglund constructs a differential graded algebra resolution of the corresponding quotient ring and computes the number of basis elements adjoined in certain multidegrees. However, in this construction Berglund commits a small error. The purpose of this section is to correct this error, and in the process, to generalize Berglund's result to the $\mathbb{D}$-graded setting. First, we define some terms and fix some notation relevant to the setting.

Notation 5.0.1. *A* monomial ideal *is an ideal $I \subset k[x_1, \ldots, x_n]$ generated by monomials. R has* monomial relations *or is a* monomial algebra *if $R = Q/I$ with I monomial. When R is monomial, it has an $\mathbb{N}^n$-grading obtained by assigning* $\deg(x_i) = e_i$*, where e_i is the i-th standard basis vector of $\mathbb{N}^n$ For $j \in \mathbb{N}^l$, the notation x^j denotes the monomial $x_1^{j_1} \cdots x_n^{j_n}$.*

More specifically, Berglund proves the following:

Theorem. *[13, Theorem 4] Let I be an ideal in $Q = k[x_1, \dots, x_n]$ and suppose I is minimally generated by a set M of square-free monomials of degree at least 2. Set $R = Q/I$, let $\alpha \in \{0,1\}^n$ and let $i \geq 2$. Then*

$$\varepsilon_{i,\alpha}(R) = \begin{cases} 0 & x^\alpha \notin \{\mathrm{lcm}(m_1, \dots, m_l) \,|\, m_1, \dots, m_l \in M\} \\ \dim_k \tilde{H}_{i-1,\alpha}(\Delta'_{M_\alpha}; k) & x^\alpha \in \{\mathrm{lcm}(m_1, \dots, m_l) \,|\, m_1, \dots, m_l \in M\} \end{cases}$$

where Δ'_{M_α} is a certain simplicial complex associated to the pair M, α. [1]

Berglund's strategy for proving this is to create a semi-free extension $Q[X]$ over $Q = k[x_1, \dots, x_n]$ based on the combinatorial data of the poset of monomials M ordered under divisibility. If this extension was both minimal rel Q and acyclic, it would be a minimal model of R by theorem 3.2.21 and so would relate all of the deviations of R to the combinatorial data of M. However, $Q[X]$ need not be minimal nor acyclic, but it is at least minimal and acyclic in degrees coming from $\mathbb{N}_{\geq 1} \times \{0,1\}^2$. Berglund's strategy is to minimize $Q[X]$, obtaining a minimal DB algebra $Q[Z]$, and then take the minimal model $Q[Y]$ of R over $Q[Z]$. Berglund argues that $Q[Y]$ is then a minimal model of R over Q, and that the above procedure does not affect the set $X_{\mathbb{N}_{\geq 1}, \{0,1\}^2}$, and hence $\varepsilon_{i,j}(R) = \# X_{i,j}$ whenever $(i,j) \in \mathbb{N}_{\geq 1} \times \{0,1\}^2$.

More specifically, Berglund makes the following claim:

Claim 1 ([12], Lemma 4)**.** *Let $Q[X]$ be an $\mathbb{N}^n$-local semifree extension with $H_0(Q[X]) = R$, and assume*

$$H_i(Q[X])_\alpha = 0$$

for all $\alpha \in \{0,1\}^n$ and $i > 0$. Then $Q[X]$ can be embedded into a semifree extension $Q[Y]$ satisfying $H(Q[Y]) \cong R$, and that $Q[Y]_\alpha = Q[X]_\alpha$ for all $\alpha \in \{0,1\}^n$.

[1]See [13, §2.2], for precise definition of Δ'_{M_α}.

Furthermore, if $Q[X]$ is minimal, then $Q[Y]$ may be chosen to be minimal.

Unfortunately the last part of this claim is false. When $Q[X]$ is minimal rel Q, the minimal model $Q[Y]$ will only be minimal rel $Q[X]$, which may be different from minimality rel Q.

Example 5.0.2. Let $|x| = 2$ and consider the semifree extension $k[x \mid \partial(x) = 0]$ where k is a field of characteristic zero. This algebra is minimal rel k, but not acyclic, since $H_2(k[x]) = kx$. The minimal model $(k[x])[y \mid \partial(y) = x]$ of $k[x]$ is minimal rel $k[x]$ and is acyclic, but is not minimal rel k because

$$\partial(y) = x \notin k + \mathfrak{m}_k(x) + (x^2) = k + (x^2)$$

The minimal model of $H_0(k[x]) \cong k$ over k is just k itself in degree zero, and since minimal models are unique up to isomorphism, there is no embedding of $k[x]$ into the minimal model.

In example 5.0.2, the problem arises because variables are mapped to zero, an unlikely choice to make when attempting to efficiently build a resolution. The following example shows the same problem arises when one begins with an inexact complex and adjoins a variable to kill a non-zero homology element, but the non-zero homology element killed is not a minimal generator of homology.

Example 5.0.3. Let $Q = k[x, y]$ and let $E = Q[e_1, e_2 \mid \partial(e_1) = x^3, \partial(e_2) = x^2 y]$. $H_1(E)$ is minimally generated by the element $ye_2 - xe_1$. Adjoin w and z in degree 2 with $\partial(w) = yxe_2 - x^2 e_1$ and $\partial(z) = ye_2 - xe_1$, obtaining $A = Q[e_1, e_2, w, z]$ which is minimal rel Q. Then the element $w - xz$ represents a nontrivial element in homology in degree 3, and adjoining a variable to kill this cycle results in the acyclic closure of A failing to be minimal over Q. The minimal model of $H_0(A)$ over Q has only a single variable in degree 2 to kill the cycle $ye_2 - xe_1$. Hence A is DB algebra which is minimal rel Q, but can not be embedded into the minimal model of $H_0(A)$ over Q.

Examples 5.0.2 and 5.0.3 above reveal what goes wrong in Berglund's proof of his Lemma 4: adjoining variables minimally to an existing DB algebra $Q[X]$ ensures only minimality rel $Q[X]$, not rel Q. In particular, if $Z(Q[X]) \nsubseteq Q + \mathfrak{m}_{Q[X]}(X) + (X^2)$, then the minimal model of $Q[X]$ is not minimal over Q.

5.1 Directed Colimits of Minimal Models

Our approach (theorem 5.1.7) to computing the multigraded deviations will be to replace the two-step procedure of minimizing and taking the minimal model with the infinite procedure of adjoining variables and minimizing, one after the other. Hence it will be necessary to prove more facts about how colimits, adjoining variables, and minimizing interact with homology and minimality in particular homological degrees. While in Berglund's original application Q was a polynomial ring and $Q \to R$ was a minimal standard presentation of a monomial ring, we will state results in their natural level of generality.

Lemma 5.1.1. *Let $\{(A^i, f_{ij})\}_{i \leq j \in I}$ be a directed system of absolutely minimal $\mathbb{D}$-local DB algebras, let $(A, f_i : A_i \to A)_{i\in I}$ be the direct colimit. Then A is absolutely minimal.*

Proof. Let $a \in A$. Obtain $i \in I$ and $b \in A_i$ such that $f_i(b) = a$. Then $\partial(a) = f_i\partial(b)$. By assumption, $\partial(b) \in \mathfrak{m}^2_{A_i}$. Since each f_i is a bigraded map of $\mathbb{D}$-local algebras, $f_i(\mathfrak{m}_{A_i}) \subset \mathfrak{m}_A$. Hence $\partial(a) = f_i(\partial(b)) \in (\mathfrak{m}^2_A)$. Since $a \in A$ was arbitrary, A is absolutely minimal. □

Lemma 5.1.2. *Let $A = Q[Y]$ be a be $\mathbb{D}$-local. Let $(i, j) \in \mathbb{N} \times \mathbb{D}$, and suppose A is absolutely minimal in degrees (i, j) and $(i + 1, j)$. Let $Q[X]$ be the minimization of $Q[Y]$. Then there is a bijection of sets $Y_{i,j} \leftrightarrow X_{i,j}$.*

Proof. Set $A/\mathfrak{m}_A = k$. By lemma 3.2.26, $\operatorname{ind} Q[Y]$ and $\operatorname{ind} Q[X]$ are quasi-isomorphic, and since $Q[X]$ is minimal rel Q, $H_{i,m}(\operatorname{ind} Q[X]) \cong kX_{i,m}$. By assumption, we have $\partial(\operatorname{ind} Q[Y]_{i,j}) = 0$ and $\partial(\operatorname{ind} Q[Y]_{i+1,j}) = 0$, so

$$kX_{i,m} \cong H_{i,m}(\operatorname{ind} Q[X]) \cong H_{i,m}(\operatorname{ind} Q[Y]) \cong kY_{i,m}.$$

□

This result extends to sets of internal degrees.

Corollary 5.1.3. *Given $D \subset \mathbb{D}$, if $Q[Y]$ is absolutely minimal in all D-degrees, then $\#Y_{i,d} = \#X_{i,d}$ for all $i \in \mathbb{N}$ and $d \in D$.*

Lemma 5.1.4. *Let $A = Q[Y]$ be $\mathbb{D}$-local with $A/\mathfrak{m}_A = k$ and let $D \subset \mathbb{D}$ be summand closed. Suppose A is absolutely minimal in all D-degrees and $H_{\geq 1}(A)_D = 0$. Then the minimization $Q[X]$ of A satisfies $H_{\geq 1}(Q[X])_D = 0$.*

Proof. If Q is characteristic zero, then Lemma 3.2.26 gives that $Q[X]$ and $Q[Y]$ are quasi-isomorphic and so the result is immediate.

Otherwise, let $\{b_\lambda\}_{\lambda \in \Lambda}$ be a k-basis for $B(\operatorname{ind} A)$ and $\{a_\lambda\}_{\lambda \in \Lambda} \subset kY$ be linearly independent with $\partial(a_\lambda) = b_\lambda$ for each λ. Set $I = (\{a_\lambda\}_{\lambda \in \Lambda} \cup \{b_\lambda\}_{\lambda \in \Lambda})$. Recall, from the proof of 3.2.26, that $Q[X] \cong Q[Y]/I$, so for each $z \in Q[Y]$, its image in $Q[X]$ will be denoted $\bar{z}$.

Let $\bar{z} \in Z(Q[X]_{i,j})$ satisfy that $\operatorname{cls}(\bar{z}) \neq 0$. Then $\partial(\bar{z}) = 0$ implies $\partial(z) \in I$ and $\bar{z} \neq 0$ implies $z \notin I$. As $\partial(z) \in I$ and $\partial(z)$ is bihomogeneous of degree $(i - 1, j)$, we may write $\partial(z) = \sum_{\lambda\in\Lambda} f_\lambda a_\lambda + g_\lambda b_\lambda$ for elements f_λ, g_λ in A, with each summand a non-zero bihomogeneous element of degree $(i-1, j)$.

First suppose some $f_\lambda a_\lambda \neq 0$. Since $a_\lambda \in Y$, it is bihomogeneous, and so f_λ is also, and $\deg(f_\lambda) + \deg(a_\lambda) = j \in D$. Since D is closed under taking summands, this requires that $\deg(a_\lambda) \in D$. As $\partial(a_\lambda) = b_\lambda \notin \mathfrak{m}_A^2$, this contradicts the minimality of A in degrees coming from D.

Next, consider the case where $g_\lambda b_\lambda \neq 0$. Then b_λ is homogeneous and so g_λ is also, and so $\deg(g_\lambda) + \deg(b_\lambda) = j \in D$. By closure under taking summands, this requires that $\deg(b_\lambda) \in D$, and hence that $\deg(a_\lambda) \in D$, contradicting the minimality of A in degrees coming from D.

Hence, in summary, any element $\bar{z}$ mapping to zero in $Q[X]$ generating a non-trivial homology element satisfies $\deg(z) \notin D$. Therefore, $H_{\geq 1}(Q[X])_D = 0$. □

Lemma 5.1.5. *Let A be any $\mathbb{D}$-graded DB algebra, and let $D \subset \mathbb{D}$ be summand-closed, and suppose that $H_{\geq 1}(A)_D = 0$. Let $A \hookrightarrow A[X]$ be any semifree extension such that $X_{\mathbb{N},D} = 0$. Then $H_{i,\underline{d}}(A[X]) = 0$ for all $(i, j) \in N \times D$.*

Proof. Suppose $z \in Z_{\geq 1}(A[X])$ is bihomogeneous with $\operatorname{cls}(z) \neq 0$. The set X is an algebra basis for $A[X]$ over A, so we may write $z = f + g$ with f and g bihomogeneous, $f \in A$, and $g \in (X)$. Since $H_{\geq 1}(A)_D = 0$, we must have $g \neq 0$. Since $g \in (X)$, $\deg(g)$ is contained in the ideal $\{\deg(x) \,|\, x \in X\} + \mathbb{D}$. By assumption, $\{\deg(x) \,|\, x \in X\} \cap D = \emptyset$, and since D is summand closed, $(\{\deg(x) \,|\, x \in X\} + \mathbb{D}) \cap D = \emptyset$ also. Hence $\deg(g) \notin D$, and hence $\deg(z) \notin D$. □

Lemma 5.1.6. *Let A be any $\mathbb{D}$-local DB algebra, let $D \subset \mathbb{D}$, and suppose that A is absolutely minimal in all D-degrees. Let $A \hookrightarrow A[X]$ be any semifree extension such that $X_{\mathbb{N},D} = 0$. Then $A[X]$ is absolutely minimal in all D-degrees.*

Proof. There is a short exact sequence of complexes of $\mathbb{D}$-graded k-vector spaces

$$0 \to \operatorname{ind} A \to \operatorname{ind} A[X] \to kX \to 0$$

and as $kX_{*,D} = 0$, $\operatorname{ind} A[X]_{*,D} = \operatorname{ind} A_{*,D}$. The absolutely minimality of A in all D-degrees means that $\operatorname{ind} A_{*,D}$ has zero differential, and so $\operatorname{ind} A[X]_{*,D}$ has zero differential. Hence $A[X]$ is minimal in all D-degrees. □

The following generalizes Berglund's Lemma (claim 1) and provides a corrected proof. We also provide several corollaries regarding long exact sequences in homotopy Lie algebras.

Theorem 5.1.7. *Let Q be a $\mathbb{D}$-local ring, and let $Q \twoheadrightarrow R$ be a surjective augmentation. Let $Q[Y]$ be a $\mathbb{D}$-local semifree extension with $H_0(Q[Y]) = R$. Let $D \subset \mathbb{D}$ be summand closed, and suppose $H_{\geq 1}(Q[Y])_D = 0$ and that $Q[Y]$ is absolutely minimal in all degrees coming from D. Then there is a minimal model $Q[X]$ of R over Q such that $\#X_{i,d} = \#Y_{i,d}$ for all $i \in \mathbb{N}$ and $d \in D$.*

Proof. Set $k = Q/\mathfrak{m}_Q$. If the characteristic of k is zero, then let $Q[\tilde{Y}]$ be a minimal model of R over $Q[Y]$ and $Q[X]$ be the minimization of $Q[\tilde{Y}]$.

By lemma 3.2.26, $Q[X]$ is quasi-isomorphic to $Q[\tilde{Y}]$ and so $Q[X]$ is acyclic and $H_0(Q[X]) \cong R$. $Q[X]$ is also minimal rel Q, and so is a minimal model of R over Q.

Since $H_{\geq 1}(Q[Y])_D = 0$ and $H_0(Q[Y]) = R$, applying lemma 5.1.5 to each step of the inductive construction of $Q[\tilde{Y}]$ gives that $\tilde{Y}_{i,d} = Y_{i,d}$ whenever $d \in D$. Since $Q[Y]$ was absolutely minimal in all D-degrees, lemma 5.1.6 gives that $Q[\tilde{Y}]$ is also. Then $\#X_{i,d} = \#\tilde{Y}_{i,d} = \#Y_{i,d}$ for all $d \in D$ by corollary 5.1.3.

When $\operatorname{char}(k) = p > 0$, we proceed inductively, starting with $A^1 = Q[Y]$. Assuming $A^{i-1} = Q[X^{i-1}]$ has been already constructed, adjoin a set of variables W^i

to minimally kill $H_{i-1}(A^{i-1})$ as in the i'th step of the minimal model construction of theorem 3.1.7, obtaining $A^{i-1}[W^i]$. Then set $A^i = Q[X^i]$ equal to the minimization of $A^{i-1}[W^i]$.

Let f_{i-1} be the composite $A^{i-1} \to A^{i-1}[W^{i-1}] \twoheadrightarrow A^i$. This defines a directed sequence $A^1 \to A^2 \ldots$ of semifree extensions of Q

We show by induction that each A^i and f_{i-1} satisfies the following:

1. $\#X^i_{j,d} = \#Y_{j,d}$ for all $j \geq 1$ and $d \in D$

2. $H_{\geq 1}(A^i)_D = 0$

3. f_{i-1} maps $X^{i-1}_{\leq i-2}$ bijectively to $X^i_{\leq i-2}$.

4. A^i is absolutely minimal

For a base case, $A^1 = Q[Y]$ satisfies $(1) - (4)$ by the hypothesis of the lemma. Assume that A^i and f_{i-1} satisfy the hypothesese above. By item 2, when W^{i+1} is adjoined to minimally kill $H_i(A^i)$, $W^{i+1}_{*,D} = \emptyset$. Then $A^i[W^{i+1}]$ is minimal in all D-degrees by lemma 5.1.6, $H_{\geq 1}(A^i[W^{i+1}])_D = 0$ by lemma 5.1.6, and $\#(X^i_{j,d} \cup W^{i+1}_{j,d}) = \#X^i_{j,d} = \#Y_{j,d}$ for all $j \in \mathbb{N}$ and $d \in D$,

The minimization $Q[X^{i+1}]$ satisfies $H_{\geq 1}(A^{i+1})_D = 0$ by lemma 5.1.4, and hence satisfies item 2. Since $A^i[W^{i+1}]$ is minimal in all $\mathbb{D}$-degrees, corollary 5.1.3 gives that $Q[X^{i+1}]$ satisfies $\#X^{i+1}_{j,d} = \#(X^i_{j,d} \cup W^{i+1}_{j,d})$ for all $j, d \in \mathbb{N} \times D$. Then by item 1, $\#X^{i+1}_{j,d} = \#Y_{i+1}$, so A^{i+1} satisfies item 1. Finally, A^{i+1} is absolutely minimal, since the minimization is always absolutely minimal by lemma 3.2.26.

Since W^{i+1} is adjoined in homological degree i, $A^i \hookrightarrow A^i[W^{i+1}]$ is bijective in homological degrees $\leq n$. Since A^i was absolutely minimal, $A^i[W^{i+1}]$ is absolutely minimal in homological degrees $1, 2 \ldots, i-1$ and in all internal degrees. Hence by lemma 5.1.2, f_i is bijective in homological degrees $1, 2, \ldots, i-2$, proving item 1.

Since the system $Q[W^{i+1}]$ satisfies item 3, lemma 2.7.2 and lemma 2.7.3 gives that the sequential colimit A is a semifree extension $Q[X]$. Since the system satisfies item 4, lemma 5.1.1 gives that $Q[X]$ is absolutely minimal. Each A^i satisfies $H_j(A^i) = 0$ for $1 \leq j \leq i$ and $H_0(A^i) = R$, and so $H_j(Q[X]) = 0$ for $j \geq 1$ and $H_0(Q[X]) = R$. Hence $Q[X]$ is a minimal model of R over Q. The set bijections of item 1 assemble into a bijection from $X_{j,d} = Y_{j,d}$ for each j, since X is the directed colimit of X^i in the category of bigraded sets. □

Applying this result to the exact sequences of indecomposables discussed in section 4.4, we get new results about long exact sequences of homotopy Lie algebras in particular degrees.

Corollary 5.1.8. *Suppose that $\mathbb{D}$ has no non-trivial units, and let $\varphi \colon R \to S$ be a $\mathbb{D}$-local homomorphism of algebras with $R_0 = k$ and $S_0 = l$ fields. Let $D \subset \mathbb{D}$ be summand-closed, and suppose that $\pi^1(F^\varphi)_j = \pi^2(F^\varphi)_j = 0$ whenever $j \in D$. Then $\pi^i(R)_j \cong \pi^i(S)_j$ for all $i \in \mathbb{N}$ and $j \in D$.*

Proof. Let $k[X]$ be a minimal model of R, $k[X] \to l[X, Y_0] \to S$ be a minimal factorization, and $l[X, Y]$ be a minimal model of S over $l[X, Y_0]$. Note that since $l[X]$ is absolutely minimal and acyclic, it is minimal in all D-degrees and satisfies $H_{\geq 1}(l[X])_D = 0$. By assumption, $\deg(y) \notin D$ for all $y \in Y_0$ and $y \in Y_1$, and so by lemma 5.1.5 and lemma 5.1.6, $l[X, Y]$ is minimal and acyclic in all D-degrees. Let $l[W]$ be a minimal model of S. Applying the theorem gives that $X_{i,j} \cup Y_{i,j} = W_{i,j}$ for each $i \in \mathbb{N}$ and $j \in D$. □

It is not terribly difficult to check the condition of the prior corollary, but the condition is stronger than necessary. It was only necessary to ensure minimality of $l[X, Y]$ in all D-degrees. Hence we have more generally:

Corollary 5.1.9. *Suppose that $\mathbb{D}$ has no non-trivial units and let $D \subset \mathbb{D}$ be summand-closed, and let $\varphi\colon R \to S$ be a $\mathbb{D}$-local homomorphism of algebras with $R_0 = k$ and $S_0 = l$ fields. Let $k[X]$ be a minimal model of R, $k[X] \to l[X, Y_0] \to S$ be a minimal factorization, and $l[X, Y]$ be a minimal model of S over $l[X, Y_0]$. If $l[X, Y]$ is minimal in all D-degrees, then $X_{i,j} \cup Y_{i,j} = W_{i,j}$.*

In the case of $\mathbb{N}^l$-graded algeras, minimization always behaves well in the set $\{0, 1\}^l$ of square-free multidegrees, as first observed by Berglund [12, Proposition 1]. Applying this to the relative context allows for deviations of R and S to always be related in square-free multidegrees.

Corollary 5.1.10. *Suppose that $\varphi\colon R \to S$ is a homomorphism of $\mathbb{N}^l$-local algebras with $R_0 = k$ and $S_0 = l$ both fields. Let $D = \{0, 1\}^l$ be the set of square-free multidegrees. Then φ induces exact sequences in homotopy Lie algebras in all D-degrees, as defined in definition 4.4.1. In other words, for each $j \in D$ there is an exact sequence*

$$\cdots \leftarrow \pi^i(R)_j \otimes_k l \leftarrow \pi^i(S)_j \leftarrow \pi^i(F^\varphi)_j \leftarrow \dots$$

Proof. Let $k[X]$ be a minimal model of R, $k[X] \to l[X, Y_0] \to S$ be a minimal factorization, and $l[X, Y]$ be the minimal model of S over $l[X, Y_0]$. Let $l[Z]$ be the minimization of $l[X, Y]$. By lemma 3.2.26, $\operatorname{ind} l[Z]$ and $l[X, Y]$ are quasi-isomorphic, so $H(\operatorname{ind} l[Z])$ can be substituted for $l[X, Y]$ in the long exact sequence in homology induced by the exact sequence

$$0 \to l[X] \to l[X, Y] \to l[Y] \to 0.$$

Since $l[Z]$ is absolutely minimal, $H(\operatorname{ind} l[Z]) \cong lZ$. Shifting and dualizing, we get the long exact sequence of $\mathbb{D}$-graded spaces

$$\cdots \leftarrow \pi^i(R) \leftarrow kZ_i \leftarrow \pi^i(F^\varphi) \leftarrow \dots .$$

By lemma 3.2.26, we have $H_{\geq 1}(l[Z])_D \cong H_{\geq 1}(l[X,Y])_D = 0$ and $l[Z]$ is absolutely minimal. Hence by theorem 5.1.7, $kZ_{*,D} \cong \pi^*(S)_D$. Substituting this isomorphism into the long exact sequence above gives that φ induces long exact sequences in homotopy Lie algebras in all squarefree degrees. □

Remark 5.1.11. The arguments in the above proof appear to apply equally well to semifree Γ-extensions. One potential implication is the computation of the k-spaces $\pi^i_\gamma(\varphi)_j$, as described in [16, p. 3.5], for particular internal degrees. We leave this exploration to future work.

Chapter 6

Off-Diagonal Deviations and the Koszul Property

In this chapter we will focus on the case of finite type $\mathbb{N}$-graded algebras whose degree 0 component is a field. Any such algebra is minimally presentable as a quotient $k[x_1, \ldots, x_n]/I$ with $I \subset (x_1, \ldots, x_n)^2$ and $\deg(x_i) \geq 1$ for each i. A well-studied case further assumes that the algebra is *quadratic*, in the sense that in a presentation as above, I is generated by elements in the k-span of the quadratic polynomials $\{x_i x_j \,|\, 1 \leq i, j \leq n\}$.

Definition 6.0.1. Let R be an $\mathbb{N}$-graded algebra with $R_0 = k$, and let $R = k[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ be a minimal presentation. R is *standard graded* if $\deg(x_i) = 1$ for each i. R is *quadratic* if $\deg(f_i) = 2$ for each i.

Quadratic algebras enjoy rich combinatorial and homological structure and duality properties; a complete reference on quadratic algebras is the book by Polischuk and Positselski [31]. A subclass of quadratic algebras of particular interest are those enjoying the *Koszul property*, of which there are many equivalent conditions. Koszul algebras are of interest because they enjoy duality theory similar to the duality between quadratic semifree extensions and Ext algebras as described in section 4.2. An explanation of this duality can be found in [16, p. 2.3]. For the purpose of this work, we will use the following definitions of the Koszul property. For an explanation of why these conditions are equivalent, see [31, Ch 1, 2]. In loc. cit. it is shown that standard graded Koszul algebras must be quadratic. Our definition will be slightly more general by allowing algebras which are not standard graded. However, the standard graded case will still be of interest, as characterizations in terms of deviations are numerically simpler.

Definition 6.0.2. Let R be an algebra of finite type over a field $k = R_0$. R is *Koszul* if the algebra $\mathrm{Ext}_R(k, k)$ under composition product is generated in homological degree 1. If R is standard graded, the following are equivalent definitions of the Koszul property:

1. $\mathrm{Ext}^i_R(k, k)_j = 0$ whenever $i \neq j$.

2. k has a linear graded free resolution over R.

Let $R\langle X\rangle$ be an acyclic closure of k over R. Then by theorem 3.2.23, $\mathrm{Hom}_R(R\langle X\rangle, k)$ has zero differential. Using the Γ-monomial basis of $R\langle X\rangle$ as a free R-module, and translating the gradings of X to statements in terms of deviations, we get the following

Proposition 6.0.3. *R is standard graded Koszul if and only if the "off-diagonal" deviations, that is, $\varepsilon_{ij}(R)$ for $i \neq j$, are all zero.*

Avramov and Peeva [10] strengthened the above result by showing that vanishing of "most" off-diagonal deviations forces the algebra to decompose as a tensor product over k of a polynomial algebra and a Koszul algebra. Furthermore, they connected this property to finite Castelnuovo-Mumford regularity of the residue field. The Castelnuovo-Mumford regularity is an important homological invariant for graded rings, and we first recall its definition.

Definition 6.0.4. Let R be an $\mathbb{N}$-graded ring and M be an $\mathbb{N}$-graded R-module. Let $\beta_{ij}(M) = \dim_k \operatorname{Tor}_i^R(M,k)_j$ be the bigraded Betti numbers of M. The *Castelnuovo-Mumford regularity* of M is

$$\operatorname{reg}_R(M) = \max\{j - i \,|\, \beta_{ij}(R) \neq 0\}.$$

Theorem 6.0.5 ([10, Theorem 2]). *The following are equivalent:*

1. $R \cong K \otimes_k S$ *with* S *a polynomial ring and* K *standard graded Koszul*
2. $\varepsilon_{ij}(R) = 0$ *whenever* $i \neq j$ *and* $i \geq 2$
3. $\operatorname{reg}_R(k) < \infty$

In particular, if R *is standard graded and* $\operatorname{reg}(k) < \infty$ *then* $\operatorname{reg}(k) = 0$ *and* R *is standard graded Koszul. More generally, when* $S = k[t_1, \ldots, t_n]$*, we have*

$$\operatorname{reg}_R(k) = \operatorname{reg}_S(k) = \sum_{i=1}^{n} \deg(t_i) - 1$$

The goal of this chapter is to extend this result to a statement regarding off-diagonal vanishing of deviations in homological degrees at least 3. We begin with some statements about how the existence of long exact sequences in homotopy Lie algebras allows for the transfer of the Koszul property.

6.1 Koszul Transfer

When R and S are $\mathbb{N}$-graded algebras over fields and $\varphi\colon R \to S$ induces an exact sequence in homotopy Lie algebras, a few facts about transfer of the Koszul property along φ follow immediately from the placement of zeros in the long exact sequence:

Theorem 6.1.1. *Let $\varphi\colon R \to S$ be a $\mathbb{D}$-local map of algebras of finite type over fields $R_0 = k$ and $S_0 = l$, and suppose φ induces a long exact sequence in homotopy Lie algebras as in definition 4.4.1:*

$$\cdots \leftarrow \pi^i(R) \otimes_k l \leftarrow \pi^i(S) \leftarrow \pi^i(F^\varphi) \leftarrow \dots$$

Then,

1. *If $\varepsilon_{ij}(\varphi) = 0$ for $j > i - 1$ and $i \geq 2$, then $\operatorname{reg}_R(k) < \infty$ if and only if $\operatorname{reg}_S(l) < \infty$.*
2. *If $\varepsilon_{ij}(\varphi) = 0$ for $j > i - 1$ and either φ is surjective or both R and S are standard graded, then $\operatorname{reg}_S(l) = \operatorname{reg}_R(k)$*
3. *If $\varepsilon_{ij}(\varphi) = 0$ for $j > i$ and $i \geq 2$, then $\operatorname{reg}_R(k) < \infty$ implies $\operatorname{reg}_S(l) < \infty$.*
4. *If $\varepsilon_{ij}(\varphi) = 0$ for $j > i$ and either φ is surjective or S is standard graded, then $\operatorname{reg}_R(k) \geq \operatorname{reg}_S(l)$.*

Proof. For item 1, the assumptions give isomorphisms $\pi^i(R)_j \cong \pi^i(S)_j$ for each $j > i \geq 2$. By theorem 6.0.5, finiteness of either $\operatorname{reg}_R(k)$ or $\operatorname{reg}_S(l)$ will hold if and only if the corresponding side of the isomorphism vanishes, from which 1) follows. For item 2, by item 1 we need only prove the equality when each of $\operatorname{reg}_S(l)$ and $\operatorname{reg}_R(k)$ are finite. In this case, by theorem 6.0.5, regularity is given by the following formulas:

$$\operatorname{reg}_R(k) = \sum_j (j-1)\varepsilon_{1j}(R), \quad \operatorname{reg}_S(l) = \sum_j (j-1)\varepsilon_{1j}(S).$$

If R and S are both standard graded, computing regularities with the formulas above shows $\operatorname{reg}_R(k) = \operatorname{reg}_S(l) = 0$. If φ is surjective, vanishing of $\varepsilon_{ij}(\varphi)$ implies

$\varepsilon_{1j}(R) = \varepsilon_{1j}(S)$ for each j and so computing regularities with the formulas above shows $\operatorname{reg}_R(k) = \operatorname{reg}_S(l)$.

For item 3, the assumptions on vanishing of $\varepsilon_{ij}(\varphi)$ give the following exact sequence for each $i \geq 2$:

$$\begin{array}{ccccc} & & 0 \longleftarrow & \pi^{i+2}(F^{\varphi})_{i+1} & \hookleftarrow \\ \pi^{i+1}(R)_{i+1} & \longleftarrow & \pi^{i+1}(S)_{i+1} & \longleftarrow \pi^{i+1}(F^{\varphi})_{i+1} & \hookleftarrow \\ \pi^{i}(R)_{i+1} & \longleftarrow & \pi^{i}(S)_{i+1} & \longleftarrow 0 & \end{array}$$

and so if $\operatorname{reg}_R(k) < \infty$, $\pi^i(R)_{i+1} = 0$ and consequently $\pi^i(S)_{i+1} = 0$. For $j > i+1$, vanishing of $\varepsilon_{ij}(\varphi)$ and $\varepsilon_{i+1,j}(\varphi)$ implies isomorphisms $\pi^i(R)_j \cong \pi^i(S)_j$, and so it follows that $\varepsilon_{ij}(S) = 0$ for $j > i \geq 2$. Hence $\operatorname{reg}_S(l) < \infty$ by [10, Theorem 2]

item 4 holds trivially if $\operatorname{reg}_R(k) = \infty$, so assume it is finite. Then item 3 gives that $\operatorname{reg}_S(l) < \infty$, and using the same formulas above to calculate regularities, it suffices to show that $\varepsilon_{1j}(R) \geq \varepsilon_{1j}(S)$ for each $j > 1$. If S is standard graded, $\varepsilon_{1j}(S) = 0$ for $j > 1$ so the statement holds trivially. If φ is surjective, then $\pi^1(F^{\varphi}) = 0$, so the long exact sequence in homotopy Lie algebras gives a graded injection $\pi^1(S) \to \pi^1(R)$, from which the inequality follows. Hence $\operatorname{reg}_R(k) > \operatorname{reg}_S(l)$ in either case.

□

Remark 6.1.2. When φ is surjective, $\varepsilon_{ij}(F^{\varphi}) = 0$ for $j > i-1$ implies that the minimal model $R[X]$ of S over R is linear. so S has a linear free resolution over R, and so φ is a *koszul morphism* as defined in [31, Chapter 2., §5]. It is known that $\varphi \colon R \to S$ is Koszul implies that R is Koszul if and only if S is Koszul [31, Chapter 2., Corollary 5.4].

The results of theorem 6.1.1 are similar to those in [31, Chapter 2., §5], but hold for maps which are not surjective. However, the assumptions of theorem 6.1.1 are stronger than those of the theorems in loc. cit., since they require properties of the minimal model of S over R. In particular, we do not know if φ Koszul implies that $\varepsilon_{ij}(F^{\varphi}) = 0$ for $j > i$.

In light of the above facts, we introduce the following definition.

Definition 6.1.3. A bigraded map $\varphi\colon R \to S$ of $\mathbb{N}$-graded algebras is *generalized Koszul of level* r if $\varepsilon_{ij}(\varphi) = 0$ whenever $j > i - 1 + r$.

Example 6.1.4. Let R be an $\mathbb{N}$-graded algebra. Let $r > 0$ and let $(f_1, \ldots, f_m)$ be a regular sequence in R with $\deg(f_j) < r$ for each j. The minimal model of S over R is the Koszul complex $R[x_1, \ldots, x_m]$ with $\deg(x_j) < r$ for each j. Hence the quotient map $R \to R/(f_1, \ldots, f_m)$ is generalized koszul of level r.

6.2 Complete Intersection Factorizations

The goal of this section is to extend theorem 6.1.1 to the case where $\varepsilon_{ij}(\varphi) = 0$ when $j > i$ and $i \geq 3$. In the absolute case, based on theorem 6.0.5, Ferraro asked the following:

Question 6.2.1. *[21, Question 3.3] If $\varepsilon_{ij}(R) = 0$ for $i \neq j$ and $i \geq 3$, then does R admit a decomposition $R \cong Q \otimes_k S$ with Q Koszul and S a complete intersection?*

The answer to this question is no. Before providing an example, we need the following fact about how deviations change when quotienting by regular elements. This recovers a theorem proven by Shamash [35] in the local case (although stated in quite different language).

Proposition 6.2.2 ([35, Theorem 1])**.** *Let R be a $\mathbb{D}$-local ring and $r \in R$ be a regular element and $S = R/(r)$.*

When $r \in \mathfrak{m}_R^2$, there are equalities

$$\varepsilon_{ij}(S) = \begin{cases} \varepsilon_{2j} + 1 & j = \deg r \\ \varepsilon_{ij} & \textit{else.} \end{cases}$$

When $r \notin \mathfrak{m}_R^2$, there are equalities

$$\varepsilon_{ij}(S) = \begin{cases} \varepsilon_{1j} - 1 & j = \deg r \\ \varepsilon_{ij} & \textit{else.} \end{cases}$$

Proof. Let $k[X]$ be a minimal model of R. Since r is a regular element in $R = H_0(k[X])$, remark 2.6.10 gives that $H(k[X, x \,|\, \partial(x) = r]) \cong S$.

When $r \in \mathfrak{m}_R^2$, $k[X, x]$ is absolutely minimal and so is the minimal model of S. The formula for the deviations then follows from the fact that $|x| = 1$ and $\deg(x) = \deg(r)$. When $r \notin \mathfrak{m}_R^2$, [8, p. 7.2.11] (a variant of remark 2.6.10) gives that $S \simeq k[X, x] \simeq k[X]/(r)$. $k[X]/(r)$ is the minimization of $k[X, x]$ rel k, and so is absolutely minimal and is the minimal model of S. After a possible linear change of variables isomorphism of $k[X_0]$, $k[X]/(r) \cong k[X']$ with $\#X'_{0j} = \#X_{0j} - 1$ when $j = \deg(r)$ and $\#X'_{ij} = \#X_{ij}$ for all other i and j. This gives the second formula for the deviations of S. □

Example 6.2.3. $R = k[x, y, z]/(xy, xz, z^2)$ is Koszul and Cohen Macaulay with depth 1. Let f be a homogeneous regular element of degree three, and set $S = R/(f)$. Since R is Koszul and f has degree 3, proposition 6.2.2 gives that $\varepsilon_{ij}(S) = 0$ when $i \neq j$ and $i \geq 3$.

The Hilbert series of $S = R/(f)$ is $(1 + t + t^3)(1 + 2t)$ and S is a zero dimensional ring. If $S \cong A \otimes_k B$, then $\dim A + \dim B = 0$ which implies $\dim A = \dim B = 0$ and the Hilbert series of A and B are polynomials dividing $(1+t+t^3)(1+2t)$.

These are irreducible, so as long as $A \neq k$ and $B \neq k$, one of A or B has Hilbert series $1+2t$. The only graded algebra with Hilbert series $1+2t$ is $C = k[a,b]/(a^2,ab,b^2)$. If $C \hookrightarrow S$, there are linear forms g_1, g_2 in S which square to zero and are linearly independent over k. But if $(ax+by+cz)^2 = 0$, $a = b = 0$, so $g_i \in \operatorname{Span}(z)$ which is a one-dimensional vector space, yielding a contradiction. Hence S can not decompose in this manner.

Finally, note that S is standard graded and $\beta^S_{2,3}(k) \neq 0$, so S is not Koszul, and also S is not a polynomial ring. Hence the desired decomposition of S does not exist.

In example 6.2.3, notice that the algebra is a quotient of a Koszul algebra by a regular sequence. In fact, this phenomenon characterizes algebras with vanishing off-diagonal above homological degree 3. The following theorem includes this fact as a special case.

Theorem 6.2.4. *Let $\varphi\colon R \to S$ be an $\mathbb{N}$-local map of graded algebras over k and l, respectively. Let $R \to R' \xrightarrow{\tilde{\varphi}} S$ be a factorization of φ, and let $f_1,\dots,f_c$ minimally generate* $\ker(R' \to S)$*, ordered by increasing degrees. Suppose that $\varepsilon_{ij}(\varphi) = 0$ for $j > i \geq d$, and suppose* $\deg(f_s) \geq d$*. Then $f_s,\dots,f_c$ is regular on $R'/(f_1,\dots,f_{s-1})$.*

Proof. Since $\pi^*(\varphi) = \pi^*(\tilde{\varphi})$, we may assume φ is surjective. We induct on the length $c-s$ of the list of minimal generators of degree at least d. When $c-s=0$, there is nothing to show.

Let $R[Y]$ be a minimal model for φ and set $S' = R/(f_1,\dots,f_{c-1})$. Then $R[Y_1]$ is the Koszul complex on $f_1,\dots,f_c$, and $\{\operatorname{cls}(\partial(y)) \mid y \in Y_2\}$ forms a minimal generating set for $H_1(R[Y_1])$. The long exact sequence in Koszul homology induced by multiplication by f_c gives that

$$H_1(R[Y_1]) \to S' \xrightarrow{f_c} S'$$

is exact. Set $K = \ker(S' \xrightarrow{f_c} S')$, so that $H_1(R[Y_1]) \to K \to 0$ is exact.

Note that since multiplication by f_c is a degree $\deg(f_c) \geq d$ map, K is contained in a graded vector space concentrated in degrees at least $\deg(f_c)$.

$$H_1(f_1, \ldots, f_c) \to K \to 0$$

and tensoring with k induces an exact sequence of graded vector spaces, inducing an inequality

$$\dim_k H_1(f_1, \ldots, f_c) \otimes_R k \geq \dim_k K \otimes_S k.$$

Continuing the inductive construction of the minimal model, $\varepsilon_{3j}(R) = \#Y_{2j}$ forms a minimal generating set of $H_1(f_1, \ldots, f_c)$. Tensoring with k yields a basis, so by assumption,

$$0 = \varepsilon_{3j}(\varphi) = \dim_k \left(H_1(f_1, \ldots, f_c) \otimes_R k\right)_j$$

for $j \geq \deg(f_c)$. Hence $\dim_k(K \otimes_S k)_j = 0$ for all j, and so $K \otimes_R k = 0$. By Nakayama's lemma, we obtain that $K = 0$.

Hence f_c is regular on S'. Let φ' be the map from R to S', and let $R[X]$ be a minimal model for S'. In particular $R[X_1]$ is the Koszul complex on $f_1, \ldots, f_{c-1}$. By proposition 6.2.2, when $f_c \in \mathfrak{m}_R^2$,

$$\varepsilon_{ij}(\varphi') = \begin{cases} \varepsilon_{ij}(\varphi) - 1 & i = 2,\, j = \deg(f_c) \\ \varepsilon_{ij}(\varphi) & \text{else} \end{cases}$$

and when $f_c \notin \mathfrak{m}_R$,

$$\varepsilon_{ij}(\varphi') = \begin{cases} \varepsilon_{ij}(\varphi) - 1 & i = 1,\, j = \deg(f_c) \\ \varepsilon_{ij}(\varphi) & \text{else} \end{cases}.$$

In either case the hypotheses on deviations are satisfied for φ'. By induction, we are done. □

Corollary 6.2.5. *Let $\varphi\colon R \to S$ be a map of graded algebras over k and l so that $\varepsilon_{ij}(\varphi) = 0$ for $j > i \geq 2$. Then φ factors as $R \xrightarrow{\alpha} T \xrightarrow{\beta} S$ with β a complete intersection homomorphism and α generalized Koszul of level* 1*.*

Proof. Let $R \to R' \xrightarrow{\widetilde{\varphi}} S$ be a minimal standard factorization of φ. Let $\ker(\widetilde{\varphi})$ be minimally generated by $f_1, \ldots, f_c$. Taking $d = 3$ in the theorem, we get a factorization $R' \xrightarrow{\widetilde{\alpha}} T \xrightarrow{\beta} S$ with β a complete intersection homomorphism generated by a regular sequence of maximal length contained in $f_1, \ldots, f_c$. By maximality, $\varepsilon_{2j}(\alpha) = 0$ when $j > 2$. Since β is a complete intersection homomorphism, $\varepsilon_{\geq 3}(\alpha) = \varepsilon_{\geq 3}(\varphi)$, from which it follows that $\widetilde{\alpha}$ is generalized Koszul of level 1.

Let α be the composition $\alpha\colon R \to R' \xrightarrow{\widetilde{\alpha}} T$. Since $R \to R' \to S$ was already a minimal standard factorization of φ, R' is flat over R and $R'/(\mathfrak{m}_R R')$ is a polynomial ring. Hence $R \to R' \xrightarrow{\widetilde{\alpha}} T$ is a minimal standard factorization of α. Therefore $\varepsilon_{ij}(\alpha) = \varepsilon_{ij}(\widetilde{\alpha})$ for all i, j. Hence α is generalized Koszul of level 1. □

Applying this result when $d = 3$ and to the minimal presentation of S as a quotient of a polynomial ring, we get the following:

Corollary 6.2.6. *Let S be an $\mathbb{N}$-local algebra with $S_0 = k$, and suppose that $\varepsilon_{ij}(S) = 0$ when $j \neq i$ and $j \geq 3$. Then $S \cong (Q \otimes_k P)/(f_1, \ldots, f_c)$ with Q standard-graded Koszul,*

P a polynomial ring, and $f_1, \dots, f_c$ a regular sequence. If S is standard graded then $S \cong Q/(f_1, \dots, f_c)$ with Q Koszul and $f_1, \dots, f_n$ a regular sequence.

Proof. Let $R \xrightarrow{\varphi} S$ be a minimal presentation of S as a quotient of a polynomial ring. Then $\varepsilon_{ij}(\varphi) = \varepsilon_{ij}(S)$, so applying the theorem gives that φ factors as $R \xrightarrow{\alpha} T \xrightarrow{\beta}$ with β a complete intersection and α generalized Koszul of level one. Since R is a polynomial ring, $\operatorname{reg}_R(k) < \infty$ and so by Proposition 6.1.1, $\operatorname{reg}_T(k) < \infty$. By theorem 6.0.5 $T \cong Q \otimes_k P$ with Q Koszul and P a polynomial ring. If S is standard graded then R can be taken to be also, from which it follows that T is Koszul, so we can take $T = Q$ and $P = k$ in the decomposition $T \cong Q \otimes_k P$. □

Chapter 7

Rigidity and Asymptotic Invariants

Few sequences of integers can be realized as the sequence of deviations of a local ring. This property is informally known as the *rigidity* of deviations. Constraints on the sequence of deviations enrich homological characterizations of different singularity types. The first result of this kind, due to Gulliksen, establishes a wide gulf between complete intersections and all other singularities.

Theorem 7.0.1 (Gulliksen, [23]). *Let R be a local ring. If $\varepsilon_i(R) = 0$ for $i \gg 0$, then $\varepsilon_i(R) = 0$ for $i \geq 3$ and R is a complete intersection.*

The strongest of these results, proven by Halperin for rings and generalized by Avramov [4] to homorphisms, shows that deviations can not vanish in even a single degree unless ring or homomorphism is a complete intersection.

Theorem 7.0.2. *Let $\varphi\colon R \to S$ be a local homomorphism such that S has a finite resolution by flat R modules. and let $R \to R' \xrightarrow{\tilde{\varphi}} \hat{S}$ be a minimal Cohen factorization of φ. If $\varepsilon_i(\varphi) = 0$ for some $i > 3$, then $\varepsilon_i(\varphi) = 0$ for all $i \geq 3$ and $\ker(\tilde{\varphi})$ is generated by a regular sequence.*

The goal of this chapter is to establish similar results for graded deviations, especially involving vanishing of graded deviations in particular internal degrees. In the case of $\mathbb{N}$-graded algebras of finite type over a field, as discussed in chapter 6, vanishing of off-diagonal deviations is related to the Koszul property. Hence, proving rigidity results for off-diagonal deviations will similarly show a substantial difference in homological behavior between Koszul and non-Koszul algebras. Throughout, we will assume that all rings are $\mathbb{N}$-graded and that their degree 0 component is a field. The arguments likely generalize to other settings, but as our primary application involves the Koszul property, we leave such generalizations to future work.

7.1 Category

A key ingredient in the proofs of classical rigidity theorems is the notion of *weak category*. The name comes from the notion of the Lusternik-Schnirelmann category of a topological space. A brief history of the development of category for DG algebras can be found in [16, §3.2]. As explained in this work, there is a sequence of numerical invariants cat_i for each $i \in \mathbb{N}$, which are nilpotency conditions stronger than finiteness of weak category. As we will only make use of weak category in this work, we will leave further exploration to the interested reader.

Definition 7.1.1. Let A be a $\mathbb{D}$-local DB algebra. The *Loewy length*, denoted $\ell\ell(A)$ of A is

$$\ell\ell(A) = \inf\{i \mid \mathfrak{m}_A^i = 0\}$$

Definition 7.1.2. Let $\varphi\colon R \to S$ be a factorizable $\mathbb{D}$-local map of k-algebras with $R_0 = k$, and let $F^\varphi = k[Y]$ be the homotopy fiber. For each i, set $k[Y_{>i}] = k[Y]/(Y_{\leq i})$. The *weak category* of φ, denoted $\mathrm{wcat}(\varphi)$, is

$$\mathrm{wcat}(\varphi) = \inf\{m \in \mathbb{N} \mid \ell\ell(H_*(k[Y_{>i}])) \leq m + 1 \text{ for all } i \geq 1\}.$$

Finite category homomorphisms arise in particular from morphisms of finite projective dimension. In particular, when $\varphi\colon k[X_0] \to R$ is a minimal presentation of R, $\mathrm{pd}_{k[X_0]}(R) < \infty$, and hence the weak category of a minimal model of R is finite. More generally, weak category is finite for any morphism of finite flat dimension.

Definition 7.1.3. Let M be a $\mathbb{D}$-graded R-module. The *flat dimension of* M, denoted $\mathrm{fl}_R(M)$, is the infinum of the lengths of resolutions of M by flat R-modules. A $\mathbb{D}$-local homomorphism $\varphi\colon R \to S$ is of finite flat dimension if $\mathrm{fl}_R(S) < \infty$.

Lemma 7.1.4. *Let $\varphi\colon R \to S$ be a bigraded $\mathbb{D}$-local homorphism satisfying $\mathrm{fl}_R(S) < \infty$. Then $\mathrm{wcat}(\varphi) \leq \mathrm{fl}_R(S) + \mathrm{edim}(S/\mathfrak{m}_R S) + 1 < \infty$.*

Proof. The local case is contained in [4]. The graded case may be recovered from the local case by localizing at $\mathfrak{m}$. The key observation is that each of $\mathrm{fl}_R(S)$, $\mathrm{edim}(S/\mathfrak{m}_R S)$ localize, as does $\ell\ell(H_*(A^{[i]}))$ for each i.

Localization of $\mathrm{fl}_R(S)$ and $\mathrm{edim}(S/\mathfrak{m}_R S)$ are classical. For Loewy length, for notational convenience set $B = A_{\mathfrak{m}_R}$ and let $\mathfrak{n}$ be its maximal ideal. Set $B^{[i]} = B/\mathfrak{n}_{\leq i}$. Notice that $(A^{[i]})_{\mathfrak{m}_R} \cong B^{[i]}$, which induces an isomorphism on homology. Hence $\ell\ell(A^{[i]})_{\mathfrak{m}_R} = \ell\ell(B^{[i]})$. □

7.2 Deviation Shifts and Rigidity

Definition 7.2.1. Let $\varphi\colon R \to S$ be an $\mathbb{N}$-local homomorphism with R_0 and S_0 both fields. The *maximal deviation shifts of* φ is the sequence

$$t_i^{\varepsilon}(\varphi) = \max\{j \mid \varepsilon_{ij}(\varphi) \neq 0\} - i.$$

When $\varphi\colon k \to R$ is the inclusion, these are the *maximal deviation shifts of* R and are denoted $t_i^{\varepsilon}(R)$.

In particular, non-vanishing of a deviation $\varepsilon_{ij}(\varphi)$ for $j > i$ exactly corresponds to the maximal deviation shift $t_i^{\varepsilon}(\varphi)$ being non-zero.

Our work will involve the construction of a particular homomorphism called a Γ-*derivation*, which we now define:

Definition 7.2.2. Let A be a DB R-algebra, let $A\langle X\rangle$ be a semifree Γ-extension, and let U be a DB module over $A\langle X\rangle$. A homological degree d R-linear map $\vartheta\colon A\langle X\rangle^{\natural} \to U^{\natural}$ is a (homological degree d) *A-linear* Γ-*derivation* if it satisfies the following axioms:

1. (A-linearity) $\vartheta(a) = 0$ whenever $a \in A$.
2. (Leibniz Rule) $\vartheta(ab) = \vartheta(a)b + (-1)^{|a|d}a\vartheta(b)$ for all $a, b \in A\langle X\rangle$.
3. (Compatibility with Divided Powers) $\vartheta(x^{(n)}) = \vartheta(x)x^{(n-1)}$ for all $x \in X$ with $|x|$ even, and all $n \in N$.

If in addition ϑ satisfies the equation $\partial\vartheta = (-1)^d\vartheta\partial$, then it is a *chain* Γ-*derivation.*

The following theorem is the main result of this chapter on the rigidity of graded deviations. In particular, it shows that once off-diagonal deviations are non-zero in some odd homological degree, they are non-zero in every remaining homological degree. The proof is similiar to that in [4], with modifications made for the inclusion of the grading.

Theorem 7.2.3. *Let $\varphi\colon R \to S$ be a graded homomorphism with $R_0 = k$ and $S_0 = l$. Assume* $\operatorname{wcat}(\varphi) < \infty$ *and let $i \geq 3$ be an odd integer. If $t_i^{\varepsilon}(\varphi) > 0$, then for each $j > i$, $t_j^{\varepsilon}(\varphi) \geq t_{j-1}^{\varepsilon}(\varphi)$.*

Proof. After possibly replacing R by R' in a minimal standard factorization $R \to R' \to S$, we may assume φ is surjective and take $R[Y]$ to be a minimal model of S.

By induction, it suffices to show that

$$t^{\varepsilon}_{i+1}(\varphi) \geq t^{\varepsilon}_{i}(\varphi) \tag{7.1}$$

$$t^{\varepsilon}_{i+2}(\varphi) \geq t^{\varepsilon}_{i+1}(\varphi). \tag{7.2}$$

Set $A = R[Y_{<i-1}]$ and let $A\langle X\rangle$ be an acyclic closure of S over A. Since $A[Y_{\geq i-1}] = R[Y]$ is a minimal model, by proposition 2.6.3 there is a quasi-isomorphism γ from $R[Y]$ to $A\langle X\rangle$. Comparing $R[Y]$ and $A\langle X\rangle$, we have:

(a) $X_j = \emptyset$ for $1 \leq j < i-2$ since $H_j(R[Y_{<i-1}]) = 0$ for $j = 1, \ldots, i-3$

(b) Y_{i-1} and X_{i-1} are in degree-preserving bijection since both count a minimal generating set of $H_{i-1}(A)$

(c) Y_i and X_i are in degree-preserving bijection since the comparison map $\gamma\colon R[Y] \to A\langle X\rangle$ is bijective in degrees $1, \ldots, 2i-1$, and so in particular $H_i(A[Y_{i-1}]) \cong H_i(A\langle X_{i-1}\rangle)$.

Set $t = t^{\varepsilon}_i(\varphi) + i$. Assuming that either (7.1) or (7.2) fail, we prove the following claim:

Claim: There exists an k-linear chain Γ-derivation θ of $k\langle X\rangle = k \otimes_A A\langle X\rangle(t)$ such that $\theta(x) = 1$ for some $x \in X_{i-1}$ and $\theta(x') = 0$ for all other $x' \in X$.

From the claim, we derive a contradiction as follows: the claim implies $\operatorname{cls}(x^{(r)}) \neq 0 \in H(k\langle X\rangle)$ for all $r \geq 0$, as if $x^{(r)} = \partial(v)$ for some v, then

$$1 = \theta^r(x^{(r)}) = \theta^r\partial(v) = \partial\theta^r(v) = 0.$$

Then $x^r = r!x^{(r)} \neq 0$ when $\operatorname{char}(k) = 0$, and $x \cdot x^{(p)} \cdots x^{(p^r)} = x^{(1+p+\cdots+p^r)} \neq 0$ when $\operatorname{char}(k) = p > 0$. Hence for each $r \in \mathbb{N}$, there exists a non-zero r-fold product of elements of $H(A\langle X\rangle)$.

The quasi-isomorphism $\gamma\colon R[Y] \to A\langle X\rangle$ induces a quasi-isomorphism of $k[Y_{\geq i-1}]$ with $k\langle X\rangle$. Then as $\operatorname{wcat}(\varphi) < \infty$, it follows from this quasi-isomorphism that $\ell\ell(H_*(k\langle X\rangle)) < \infty$, and taking r in the multiplication formulas above to be greater than $\operatorname{wcat}(\varphi)$ yields a contradiction. So, it suffices to prove the claim.

First assume (7.1) fails. Set $t = t_i^{\varepsilon}(\varphi) + i$. The assumptions on vanishing of deviations gives that $Y_{i-1,t} \neq \emptyset$ and $Y_{i,>t} = \emptyset$, and so section 7.2 and section 7.2 give that $X_{i-1,t} \neq \emptyset$ and $X_{i,>t} = \emptyset$.

Recall from definition 3.2.17 the complex $\operatorname{ind}^{\gamma} A\langle X\rangle$ has the form

$$\cdots \to SX_{i+1} \to SX_i \to SX_{i-1} \to 0 \to \dots$$

and that this complex is minimal by theorem 3.2.22. Choose $x \in X_{i-1,t}$. As $A\langle X\rangle$ is a graded resolution, the differential of $\operatorname{ind}^{\gamma} A\langle X\rangle$ maps X_{ij} into $(S_{\geq 1}X_{i-1})_j$, where

$$(S_{\geq 1}X_{i-1})_j = \{s_1x_1 + \cdots + s_nx_n \mid s_m \in S_{\geq 1},\, x_m \in X_{i-1},\, \deg(s_m) + \deg(x_m) = j\}.$$

Since each s_m appearing in the sum above is of positive degree, the x_m's appearing in the sum are of degree at most $j-1$. Since $X_{i,>t} = \emptyset$, applying the above fact for each j yields that that the image of the differential is contained in $SX_{i,<t}$. Hence Sx generates a free summand of the cokernel of the differential. By [8, p. 6.3.6], a homological degree $|x| = i-1$ derivation $\vartheta\colon A\langle X\rangle \to A\langle X\rangle(t)$ exists satisfying that $\vartheta(x) = 1$ and $\vartheta(x') \subset \mathfrak{m}_A A\langle X\rangle$ for $x' \neq x$. Then $\theta := l \otimes_A \vartheta$ satisfies the requirements of the claim.

Now assume (7.2) fails. Again, let $t = t_i^{\varepsilon}(\varphi) + i$. Since (7.1) holds, we have $X_{i-1,t} \neq \emptyset$ and $X_{i,>t} \neq \emptyset$. Again chose $x \in X_{i-1,t}$ and set $X'_{i-1} = X_{i-1} \setminus \{x\}$. Let

$s_1, \ldots, s_e$ be a set of minimal generators for the maximal ideal of S, and let M be the complex in the bottom row of the diagram:

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & SX_{i+1} & \xrightarrow{\partial_{i+1}} & SX_i & \xrightarrow{\partial_i} & Sx \oplus SX'_{i-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_{i+1}} & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i-1}} & & \\
\ldots & \longrightarrow & 0 & \longrightarrow & \bigoplus_{l=1}^{e} Sm_l & \xrightarrow{\left[\begin{smallmatrix} s_1 & \ldots & s_e \end{smallmatrix}\right]} & S(-t) & \longrightarrow & 0
\end{array}$$

where $m_1, \ldots m_l$ is a basis of M_i with $\deg(m_l) = \deg(s_l) - t$. The chain map $f_\bullet$ is constructed as follows:

- f_{i-1} is the map sending x to 1 and all of X'_{i-1} to zero.

- $\operatorname{ind}^\gamma A\langle X\rangle$ and M are minimal complexes, so a map f_i exists so that $\left[\begin{smallmatrix} s_1 & \cdots & s_e \end{smallmatrix}\right] f_i = f_{i-1}\partial_i$. In more detail, for each $x' \in X_i$, choose $b \in SX'_{i-1}$ and $a_1, \ldots a_e \in S$ and write
$$\partial_i(x') = b + \sum_{l=1}^{e} a_l s_l x \tag{7.3}$$
where $b \in SX'_{i-1}$. Then f_i may be defined by sending x' to $\begin{bmatrix} a_1 \\ \vdots \\ a_e \end{bmatrix}$.

- Write $SX_i = SX_{i,\leq t} \oplus SX_{i,>t}$. For each $x' \in X_{i,\leq t}$, degree considerations enforce that $a_l = 0$ for each l in equation 7.3, and hence $f_i(SX_{i,\leq t}) = 0$. Since 7.2 fails and since $\operatorname{ind}^\gamma A\langle X\rangle$ is a minimal complex, ∂_{i+1} maps X_{i+1} into $SX_{i,\leq t}$. Therefore the left square commutes.

Let U be the DG module over $A\langle X\rangle$ whose underlying module structure is free over $A\langle X\rangle^\natural$ with basis $1, u_1, \ldots, u_e$ with $\deg(1) = -t$, $\deg(u_l) = \deg(m_l)$, and with differential given by $\partial(u_l) = s_l$ for each l. The augmentation $A\langle X\rangle \to S$ induces a quasi-isomorphism from U to $U \otimes_{A\langle X\rangle} S = M$, and a derivation $\vartheta\colon A\langle X\rangle \to U$ satisfying $\vartheta(x) = 1$ and $\vartheta(x') \subset \mathfrak{m}_A A\langle X\rangle$ for $x' \neq x$.

The module $k \otimes_A U$ maps each u_l to zero and so the projection $k \otimes_A U \to k\langle X\rangle$ is a chain map. The composition of $k \otimes_A \vartheta$ with the projection is a derivation satisfying the properties required in the claim.

□

Corollary 7.2.4. *Let R be an $\mathbb{N}$-local algebra with $R_0 = k$. If $\varepsilon_{ij}(R) = 0$ for $i \neq j$ and $i \gg 0$, then $\varepsilon_{ij}(R) = 0$ for $i \neq j$ and all odd $i \geq 3$.*

Proof. By contrapositive: the theorem gives that whenever $t_i^{\varepsilon}(R) > 0$ for some $i \geq 3$, $t_n^{\varepsilon}(R) > 0$ for all $n > i$. □

It is not evident how proof of theorem 7.2.3 may be modified to remove the reference to parity in the above result. A counterexample to such a strengthened statement is unknown to the author, but would require a different approach.

Question 7.2.5. *Let R be an $\mathbb{N}$-local algebra with $R_0 = k$. If $\varepsilon_{ij}(R) = 0$ for $i \neq j$ and $i \gg 0$, then must $\varepsilon_{ij}(R) = 0$ for $i \neq j$ and all even $i \geq 3$?*

7.3 Slope

We now discuss the implications theorem 7.2.3 have on the slope of k as a module over a graded algebra R. Slope may be generally defined for any module over a graded algebra. It measures the rate of growth of the internal degree of a minimal free resolution of a module. More formally:

Definition 7.3.1. Let R be a graded algebra and M be a graded R-module. The *i-th maximal shift* of M is

$$t_i(M) = \sup\{j \mid \operatorname{Tor}_i(M,k)_j \neq 0\} - i.$$

The *slope* of M is

$$\operatorname{slope}_R M = \sup\left\{\frac{t_i(M) - t_0(M)}{i} \mid i \in \mathbb{N}\right\}.$$

As the following example demonstrates, it is the minimal slope needed so that when line of slope $\operatorname{slope}_R M$ drawn on the "compressed bigraded Betti table" of M (the table whose entries are $\beta_{i,j-i}(R)$), all the non-zero entries lie above the line.

Example 7.3.2. Let R be an $\mathbb{N}$-graded k-algebra and let a, b be a regular sequence in R_2. Then $R/(a,b)$ is resolved by the Koszul complex $R\langle x, y \mid \partial(x) = a, \partial(y) = b\rangle$, and the compressed Betti table is

j-i \ i	0	1	2
0	1	.	.
1	.	2	.
2	.	.	1

The slope of $R/(a,b)$ over R is therefore 1.

Remark 7.3.3. The definition above and an accompanying discussion may be found in [30]. A related invariant called *rate* was first defined by Backelin [11] for the shift of the maximal ideal, and generalized and shown to be finite by Aramova, Bărcănescu, and Herzog [1].

Since the supremum in definition 7.3.1 is indexed over every homological degree, its values are influenced by the initial segment $\{t_1(M), \ldots, t_l(M)\}$. Motivated by a question of Conca [30, p. 9.4], we define a numerical invariant which measures the asymptotic growth of the shifts of M.

Definition 7.3.4. Let M be a graded R module over a graded algebra R. The *limslope* of M is

$$\operatorname{limslope}_R M = \limsup \left\{ \frac{t_i(M) - t_0(M)}{i} \mid i \in \mathbb{N} \right\}.$$

One useful property about limslope is that it is stable after taking syzygies.

Proposition 7.3.5. *Let N be a j-th syzygy of M. Then* $\operatorname{limslope}_R(N) = \operatorname{limslope}(M)$.

Proof. For convenience, set $s_i(M) = t_i(M) + i$. We have

$$\frac{s_i(\Omega_j M) - i - t_0(\Omega_j M)}{i} = \frac{s_{i+j}(M) - (i+j) - (t_j(M) - j)}{i+j} \frac{i+j}{i}.$$

Taking $\limsup_{i\to\infty}$ on the lefthand side yields $\operatorname{limslope}_R(N)$ by definition. Since $\lim_{i\to\infty} \frac{i+j}{i} = 1$ exists and is finite, we have

$$\begin{aligned} &\limsup_{i\to\infty} \left(\frac{s_{i+j}(M) - (i+j) - (s_j(M) - j)}{i+j} \frac{i+j}{i} \right) \\ &= \limsup_{i\to\infty} \left(\frac{s_{i+j}(M) - (i+j) - (s_j(M) - j)}{i+j} \right) \limsup_{i\to\infty} \frac{i+j}{i} \end{aligned}$$

which equals $\operatorname{limslope}_R(M)$ by definition. □

Using deviations and theorem 7.2.3, we bound the difference between the slope and the limslope of the residue field of an algebra:

Theorem 7.3.6. *Let R be a graded k-algebra. Then either* $\operatorname{limslope}_R(k) = \operatorname{slope}_R(k)$ *or the following hold:*

1. *There exists an odd integer l so that* $\operatorname{slope}_R(k) = (t_l(k) - l)/l$. *In other words, the slope is attained in some degree l.*

2. *For any l satisfying (1), we have* $\operatorname{slope}_R(k) > \operatorname{limslope}_R(k) \geq (t_l(k) - l)/(l+1)$.

Proof. Suppose that $\operatorname{limslope}_R(k) \neq \operatorname{slope}_R(k)$. Then there exists some $n \gg 0$ so that

$$\sup_{i>n} \left\{ \frac{t_i(k) - i}{i} \right\} < \sup_{i \geq 1} \left\{ \frac{t_i(k) - i}{i} \right\} \tag{7.4}$$

and so the slope has to be attained in some homological degree j between 1 and n.

Let $R\langle X \rangle$ be an acyclic closure of k. Let $m = x_1^{(e_1)} \cdots x_n^{(e_n)}$ be a normal Γ-monomial with homological degree j and internal degree $t_j(k)$. Then we have

$$\frac{t_j(k)}{j} = \frac{\deg(m)}{|m|} = \frac{e_1 \deg(x_1) + \cdots + e_n \deg(x_n)}{e_1|x_1| + \cdots + e_n|x_n|}.$$

Choosing $x = x_l$ so that $\frac{\deg(x) - |x|}{|x|}$ is as large as possible, we must have

$$\frac{\deg(x) - |x|}{|x|} \geq \frac{t_j(k) - k}{j}$$

and so equality must hold since the slope is attained at j.

If $|x|$ is even then $\{x^{(e)}\}_{e \geq 1}$ is a sequence of elements with

$$\frac{\deg(x^{(e)}) - |x^{(e)}|}{|x^{(e)}|} = \frac{\deg(x) - |x|}{|x|} = \operatorname{slope}_R(k)$$

which contradicts eq. (7.4). Hence $|x|$ is odd.

By theorem 7.2.3, there exists a variable y with $|y| = |x| + 1$ and $\deg(y) \geq \deg(x) + 1$. The inequality

$$\frac{\deg(y) - |y|}{|y|} \geq \frac{\deg(x) + 1 - |x| - 1}{|x| + 1} = \frac{\deg(x) - |x|}{|x| + 1}$$

applied to the sequence $\{y^{(e)}_{e\geq 1}\}$ forces the inequality in item 2. □

We are unsure if this result can be strengthened to show equality in all cases. We also do not know when this result extends to calculating $\text{slope}_R(S)$ for a $\mathbb{D}$-local homomorphism $\varphi\colon R \to S$.

Question 7.3.7. *Let $\varphi\colon R \to S$ be a $\mathbb{D}$-local homomorphism with $R_0 = k$ and $S_0 = l$ fields. Does* $\text{limslope}_R(k) = \text{slope}_R(k)$*? Under what conditions does* $\text{limslope}_R(S) = \text{slope}_R(S)$*?*

Bibliography

[1] Anneta Aramova, Şerban Bărcănescu, and Jürgen Herog. "On the rate of relative Veronese submodules". In: *Rev. Roumaine Math. Pures Appl.* 3-4 (40 1995), pp. 243–251.

[2] E. Assmus. "On the Homology of Local Rings". In: *Illinois Journal of Mathematics* 3 (1950), pp. 187–199.

[3] L. Avramov. "Local algebra and rational homotopy". In: *Homotopie algeébrique et algeébre locale* 113-114 (1982).

[4] L. Avramov. "Locally Complete Intersection Homomorphisms and a Conjecture of Quillen on the Vanishing of Cotangent Homology". In: *Annals of Mathematics* 150.2 (1999), pp. 455–487. ISSN: 0003486X. URL: `http://www.jstor.org/stable/121087`.

[5] Luchezar Avramov and Stephen Halperin. "On the non-vanishing of cotangent cohomology". In: *Commentarii Mathematici Helvetici* 62.1 (Dec. 1987), pp. 169–184. ISSN: 1420-8946. DOI: `10.1007/BF02564444`. URL: `https://doi.org/10.1007/BF02564444`.

[6] Luchezar Avramov and Stephen Halperin. "Through the looking glass: a dictionary between rational homotopy theory and local algebra". In: *Algebra, algebraic topology and their interactions (Stockholm, 1983).* Vol. 1183. Lecture

Notes in Math. Springer, Berlin, 1986, pp. 1–27. DOI: 10.1007/BFb0075446. URL: https://doi-org.libproxy.unl.edu/10.1007/BFb0075446.

[7] Luchezar L. Avramov. "Homotopy Lie algebras and Poincaré series of algebras with monomial relations". In: vol. 4. 2, part 1. The Roos Festschrift volume, 1. 2002, pp. 17–27. DOI: 10.4310/hha.2002.v4.n2.a2. URL: https://doi-org.libproxy.unl.edu/10.4310/hha.2002.v4.n2.a2.

[8] Luchezar L. Avramov. "Infinite Free Resolutions". In: *Six Lectures on Commutative Algebra*. Ed. by J. Elias et al. Basel: Birkhäuser Basel, 1998, pp. 1–118. ISBN: 978-3-0346-0329-4. DOI: 10.1007/978-3-0346-0329-4_1. URL: https://doi.org/10.1007/978-3-0346-0329-4_1.

[9] Luchezar L. Avramov, Hans-Bjørn Foxby, and Bernd Herzog. "Structure of local homomorphisms". In: *J. Algebra* 164.1 (1994), pp. 124–145. ISSN: 0021-8693. DOI: 10.1006/jabr.1994.1057. URL: https://doi.org/10.1006/jabr.1994.1057.

[10] Luchezar L. Avramov and Irena Peeva. "Finite Regularity and Koszul Algebras". In: *American Journal of Mathematics* 123.2 (2001), pp. 275–281. ISSN: 00029327, 10806377. URL: http://www.jstor.org/stable/25099057.

[11] Jörgen Backelin. "On the rates of growth of the homologies of Veronese subrings". In: *Algebra, Algebraic Topology and their Interactions*. Ed. by Jan-Erik Roos. Berlin, Heidelberg: Springer Berlin Heidelberg, 1986, pp. 79–100. ISBN: 978-3-540-39790-8.

[12] A. Berglund. "Poincaré series and homotopy Lie algebras of monomial rings". In: *Research Reports in Mathematics* 6 (2005).

[13] Berglund, A. "Poincaré series of monomial rings". In: *Journal of Algebra* 295 (2006).

[14] Pierre Berthelot and Arthur Ogus. *Notes on Crystalline Cohomology. (MN-21).* Princeton University Press, 1978.

[15] N. Bourbaki. *Elements of Mathematics, Algebra I.* Springer, 1989.

[16] Benjamin Briggs. "Local Commutative Algebra and Hochschild Cohomology Through the Lens of Koszul Duality". PhD thesis. University of Toronto, 2018.

[17] Benjamin Briggs. "Vasconcelos' conjecture on the conormal module". In: *Inventiones mathematicae* 227 (2022), pp. 415–428. DOI: https://doi.org/10.1007/s00222-021-01070-0.

[18] Winfried Bruns and Jürgen Herzog. *Cohen-Macaulay Rings.* Cambridge Studies in Advanced Mathematics, 1993.

[19] David Eisenbud. *Commutative Algebra with a View Towards Algebraic Geometry.* Springer, 1995.

[20] Yves Félix et al. "The radical of the homotopy Lie algebra". In: *Amer. J. Math.* 110.2 (1988), pp. 301–322. ISSN: 0002-9327. DOI: 10.2307/2374504. URL: https://doi.org/10.2307/2374504.

[21] Luigi Ferraro. "Modules of infinite regularity over commutative graded rings". In: *Proc. Amer. Math. Soc.* 147.5 (2019), pp. 1929–1939. ISSN: 0002-9939. DOI: 10.1090/proc/14385. URL: https://doi-org.libproxy.unl.edu/10.1090/proc/14385.

[22] T. Gulliksen and G. Levin. *Homology of Local Rings.* Queen's Papers in Pure and Applied Mathematics, 1969.

[23] T. H. Gulliksen. "A homological characterization of local complete intersections". In: *Compositio Math.* 23 (1971), pp. 251–255. ISSN: 0010-437X.

[24] T. H. Gulliksen. “A proof of the existence of minimal algebra resolutions”. In: *Acta Math.* 120 (1968), pp. 53–58.

[25] Stephen Halperin. “The nonvanishing of the deviations of a local ring”. In: *Comment. Math. Helv.* 62.4 (1987), pp. 646–653. ISSN: 0010-2571. DOI: 10.1007/BF02564468. URL: https://doi-org.libproxy.unl.edu/10.1007/BF02564468.

[26] Brian Johnson. “Commutative Rings Graded by Abelian Groups”. PhD thesis. University of Nebraska Lincoln, 2012.

[27] Bernhard Keller. “On Differntial Graded Categories”. In: *International Congress of Mathematicians* 2 (2006).

[28] Huishi Li. “On Monoid Graded Local Rings”. In: *Journal of Pure and Applied Algebra* 216 (12 2012), pp. 2697–2708.

[29] Yu. I. Manin. “Quantum groups and non-commutative differential geometry”. In: *Mathematical physics, X (Leipzig, 1991)*. Springer, Berlin, 1992, pp. 113–122.

[30] Jason McCullough and Irena Peeva. “Commutative Algebra and Noncommutative Algebraic Geometry, I”. In: vol. 67. MSRI Publications, 2015. Chap. Infinite graded free resolutions.

[31] Alexander Polishchuk and Leonid Positselski. *Quadratic Algebras*. American Mathematical Society, 2005.

[32] Daniel Quillen. “Rational homotopy theory”. In: *Ann. of Math. (2)* 90 (1969), pp. 205–295. ISSN: 0003-486X. DOI: 10.2307/1970725. URL: https://doi.org/10.2307/1970725.

[33] Emily Riehl. *Categories in Context*. Dover, 2016.

[34] C. Schoeller. "Homologie des anneaux locaux noeth´eriens". In: *C.R. Acad. Sci. Paris S'er* A 265 (1967), pp. 768–771.

[35] Jack Shamash. "The Poincaré series of a local ring". In: *J. Algebra* 12 (1969), pp. 453–470. ISSN: 0021-8693. DOI: 10.1016/0021-8693(69)90023-4. URL: https://doi-org.libproxy.unl.edu/10.1016/0021-8693(69)90023-4.

[36] Gunnar Sjödin. "A set of generators for $\mathrm{Ext}_R(k,k)$". In: *Math. Scand.* 38.2 (1976), pp. 199–210. ISSN: 0025-5521. DOI: 10.7146/math.scand.a-11629. URL: https://doi-org.libproxy.unl.edu/10.7146/math.scand.a-11629.

[37] John Tate. "Homology of Noetherian rings and local rings". In: *Illinois J. Math.* 1 (1957), pp. 14–27. ISSN: 0019-2082. URL: http://projecteuclid.org/euclid.ijm/1255378502.

[38] Charles A. Weibel. "History of homological algebra". In: *History of topology*. North-Holland, Amsterdam, 1999, pp. 797–836. DOI: 10.1016/B978-044482375-5/50029-8. URL: https://doi-org.libproxy.unl.edu/10.1016/B978-044482375-5/50029-8.