FRACTAL BEHAVIOR OF THE FIBONOMIAL TRIANGLE MODULO PRIME p, WHERE THE RANK OF APPARITION OF p IS $p+1$

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ABSTRACT. Pascal's triangle is known to exhibit fractal behavior modulo prime numbers. We tackle the analogous notion in the fibonomial triangle modulo prime p with the rank of apparition $p^* = p+1$, proving that these objects form a structure similar to the Sierpinski Gasket. Within a large triangle of p^*p^{m+1} many rows, in the i^{th} triangle from the top and the j^{th} triangle from the left, $\binom{n+ip^*p^m}{k+jp^*p^m}_F$ is divisible by p if and only if $\binom{n}{k}_F$ is divisible by p. This proves the existence of the recurring triangles of zeroes which are the principal component of the Sierpinski Gasket. The exact congruence classes follow the relationship $\binom{n+ip^*p^m}{k+jp^*p^m}_F \equiv_p (-1)^{ik-nj}\binom{i}{j}\binom{n}{k}_F$, where $0 \le n, k < p^*p^m$.

1. Introduction

Pascal's triangle is known to exhibit fractal behavior modulo prime numbers. This can be proven by using Lucas' Theorem:

Theorem 1.1. *Write n and k in base p with digits $n_0, n_1, \dots, n_m$ and $k_0, k_1, \dots, k_m$. Then,*

$$\binom{n}{k} \equiv_p \binom{n_0}{k_0}\binom{n_1}{k_1}\cdots\binom{n_m}{k_m}.$$

Consider the Fibonacci numbers as defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \ge 2$. The fibonomial triangle is formed using the fibotorial $!_F$ function in place of the factorial function, where $n!_F = F_nF_{n-1}F_{n-2}\dots F_1$. Then the fibonomial coefficient $\binom{n}{k}_F$ is defined as $\frac{n!_F}{(n-k)!_F k!_F}$, where $\binom{n}{0}_F$ is defined to be 1 for $n \ge 0$, as with binomial coefficients. The fibonomial triangle appears to exhibit a fractal structure, but unfortunately Lucas' Theorem does not apply to fibonomial coefficients [7]. Instead, we prove an analogue of Lucas' Theorem for divisibility by p in section 3 and deal with exact congruence classes in section 4.

2. Background

We define p^* to be the rank of apparition of p in the Fibonacci sequence. The rank of apparition is the index of the first Fibonacci number divisible by p.

The Fibonacci sequence exhibits a number of interesting properties, among them the divisibility property, regular divisibility by a prime, and the shifting property. The following lemmas can be found in a variety of sources, including [8].

Lemma 2.1. (Lucas [5]) *For positive integers n and m, $gcd(F_n, F_m) = F_{gcd(n,m)}$. If $n \mid m$ then $gcd(n,m) = n$, so $gcd(F_n, F_m) = F_n$, and so $F_n \mid F_m$.*

Lemma 2.2. *For positive integer i and prime p, $p \mid F_{ip^*}$*

The periodic nature of the Fibonacci sequence modulo p follows.

Lemma 2.3. *For positive integers n and m, $F_{n+m} = F_mF_{n+1} + F_{m-1}F_n$.*

By a result of Sagan and Savage [6], the fibonomial coefficients have a combinatorial interpretation. It follows that $\binom{n}{k}_F$ is a nonnegative integer.

The fibonomial coefficients conform to a recurrence relation analogous to the recurrence relation on binomial coefficients:

Lemma 2.4. *For positive integers n and k,*

$$\binom{n}{k}_F = F_{n-k+1}\binom{n-1}{k-1}_F + F_{k-1}\binom{n-1}{k}_F.$$

Like the binomial coefficients, the fibonomial coefficients possess a number of useful properties, among them the negation property and the iterative property:

Lemma 2.5. (Gould [2]) *For $n, k \in \mathbb{Z}$,*

$$\binom{n}{k}_F = \binom{n}{n-k}_F.$$

Lemma 2.6. (Gould [2]) *For $a, b, c \in \mathbb{Z}$,*

$$\binom{a}{b}_F\binom{b}{c}_F = \binom{a}{c}_F\binom{a-c}{a-b}_F.$$

It is a commonly known fact that the Fibonacci sequence modulo an integer is periodic. The period modulo p is called the Pisano period and is denoted $\pi(p)$. A related notion is the Pisano semiperiod, defined as the period of the modulo p Fibonacci sequence up to a sign.

3. Divisibility

By a result of Harris [3], when $p^* = p+1$, $\pi(p) \mid 2p^*$, and p^* is the Pisano semiperiod. We rely on the notion of the semiperiod and assume for this paper that p is an odd prime and $p^* = p+1$.

For a nonnegative integer x, $\nu_p(x)$ denotes the p-adic valuation of x, i.e. the highest power of p dividing x.

We use a result of Knuth and Wilf [4], adapted for the Fibonacci sequence.

Theorem 3.1. (Knuth and Wilf) *The highest power of an odd prime p that divides the fibonomial coefficient $\binom{n}{k}_F$ is the number of carries that occur to the left of the radix point when k/p^* is added to $(n-k)/p^*$ in p-ary notation, plus the p-adic valuation $\nu_p(F_{p^*}) = 1$ if a carry occurs across the radix point.*

We require that p not be a Wall-Sun-Sun prime in order for $\nu_p(F_p^*) = 1$ to hold.

Since we are interested in divisibility, we only require that the p-adic valuation is at least one, so it suffices to show that a carry occurs.

As in [1] and [7], we consider the base $\mathcal{F}_{p^*} = (1, p^*, p^*p, p^*p^2, \dots)$. So $n = n_0 + n_1p^* + n_2p^*p + \cdots + n_mp^*p^{m-1} = (n_0, n_1, n_2, \dots, n_m)_{\mathcal{F}_{p^*}}$. In this base, division by p^* results in a number $(n_1, n_2, n_3, \dots, n_m)_p$, with fractional part $\frac{n_0}{p^*}$ only, which simplifies the counting of the carries.

Generalizing Southwick's proof in [7], we prove the following:

Theorem 3.2. *Given integers $n, k > 0$,*

$$p \mid \binom{n}{k}_F \iff p \mid \binom{n_0}{k_0}_F\binom{n_1}{k_1}_F \cdots \binom{n_m}{k_m}_F.$$

Proof. By Theorem 3.1, $p \mid \binom{n}{k}_F$ if and only if a carry occurs in the addition of $(\frac{k}{p^*})$ and $(\frac{n-k}{p^*})$ in base p. The first carry occurs either across the radix point or to the left of the radix point. Let $q = n-k = (q_0, q_1, \dots, q_m)_{\mathcal{F}_{p^*}}$

(1) First consider the conditions necessary for the carry across the radix point.

If $n_0 \geq k_0$, then $q_0 = n_0 - k_0$, $k_0 + (q_0) = n_0 < p^*$. In this case, there will be no carry.

Alternatively, if $k_0 > n_0$, then a borrow occurs, so $q_0 = n_0 - k_0 + p^*$. The addition of $\frac{k_0}{p^*}$ and $\frac{q_0}{p^*}$ produces:

$$\frac{k_0 + n_0 - k_0 + p^*}{p^*} = \frac{n_0 + p^*}{p^*} \geq 1.$$

Thus a carry occurs across the radix point if and only if $k_0 > n_0$.

(2) If a carry across the radix point does not occur, then let the first carry occur in the $(j+1)^{st}$ digit, that is, in the addition of k_j with q_j (note that the $(j+1)^{st}$ digit of n in base $\mathcal{F}_{p^*}$ is $n_j p^* p^{j-1}$). The division by p^* moves the digits to the right by one, so the carry occurs at the j^{th} digit in base p.

If $n_j \geq k_j$, then $k_j + q_j = k_j + (n_j - k_j) < p$ since we assume there was no previous carry.

If $k_j > n_j$, the subtraction $n_j - k_j$ results in a borrow, so $q_j = n_j - k_j + p$, and so

$$k_j + q_j = k_j + (n_j - k_j + p) = n_j + p \geq p.$$

This case is the only case in which a carry occurs.

Therefore if a carry occurs in the j^{th} position, then $n_j < k_j$, and so $p \mid \binom{n_j}{k_j}_F$ since $\binom{n_j}{k_j} = 0$, and so $p \mid \binom{n_0}{k_0}_F \binom{n_1}{k_1}_F \cdots \binom{n_m}{k_m}_F$. Using the above result and Theorem 3.1, we conclude that if $p \mid \binom{n}{k}_F$ then $p \mid \binom{n_0}{k_0}_F \binom{n_1}{k_1}_F \cdots \binom{n_m}{k_m}_F$.

For the reverse direction we note that all these steps and Theorem 3.1 are reversible.

Note that for $n < k$ the statement follows trivially.

Therefore $p \mid \binom{n}{k}_F \Leftrightarrow p \mid \binom{n_0}{k_0}_F \binom{n_1}{k_1}_F \cdots \binom{n_m}{k_m}_F$. □

Corollary 3.3. *Given $0 \leq m$ and $0 \leq n,\ k < p^* p^m$, $\forall i, j \in \mathbb{Z}$ such that $0 \leq j < i < p$,*

$$p \mid \binom{n + ip^*p^m}{k + jp^*p^m}_F \iff p \mid \binom{n}{k}_F.$$

Proof. By Theorem 3.2, since $p \nmid \binom{i}{j}_F$,

$$p \mid \binom{n}{k}_F \iff p \mid \binom{n_0}{k_0}_F \binom{n_1}{k_1}_F \cdots \binom{n_m}{k_m}_F \iff p \mid \binom{n_0}{k_0}_F \binom{n_1}{k_1}_F \cdots \binom{n_m}{k_m}_F \binom{i}{j}_F \iff p \mid \binom{n+ip^*p^m}{k+jp^*p^m}_F.$$ □

4. Exact Nonzero Congruence Classes

We begin with a number of necessary Lemmas.

Lemma 4.1. *If $\frac{a}{b}$, $\frac{a}{c}$, $\frac{b}{c} \in \mathbb{Z}$, with $\frac{a}{c} \equiv_p a'$ and $\frac{b}{c} \equiv_p b' \not\equiv_p 0$, then $\frac{a}{b} \equiv_p a'(b')^{-1}$.*

Proof. Since $c \neq 0$, we can multiply the fraction $\frac{a}{b}$ by $\frac{1/c}{1/c}$. Since the resulting fraction is an integer, it can be reduced modulo p to $a'(b')^{-1}$. Note that $(b')^{-1}$ exists because p is prime and $b' \not\equiv_p 0$. □

Lemma 4.2. *For $0 \leq n < p^*p^m$, $F_{n+p^*p^m} \equiv_p -F_n$.*

Proof. Since $p^* = \frac{1}{2}\pi(p)$ is the semiperiod, $F_{n+p^*} \equiv_p -F_n$.

Then since $(p^m - 1)$ is even and $\pi(p) = 2p^*$, $F_{n+p^*} \equiv_p -F_n$ implies $F_{n-p^*p^m} \equiv_p -F_n$.

□

Lemma 4.3. *For $i > 0$,*

$$\frac{F_{ip^*p^m}}{F_{p^*p^m}} \equiv_p i(-1)^{i-1}.$$

Proof. We prove this by induction.

First, let $i = 1$. Then the statement follows trivially.

Now, assume the inductive hypothesis:

$$\frac{F_{ip^*p^m}}{F_{p^*p^m}} \equiv_p (-1)^{i-1} i.$$

Consider

$$\frac{F_{(1+i)p^*p^m}}{F_{p^*p^m}} = \frac{F_{p^*p^m + ip^*p^m}}{F_{p^*p^m}}.$$

We apply the shifting property of the Fibonacci sequence to obtain:

$$\frac{F_{p^*p^m + ip^*p^m}}{F_{p^*p^m}} = \frac{F_{p^*p^m}F_{ip^*p^m+1} + F_{p^*p^m-1}F_{ip^*p^m}}{F_{p^*p^m}}.$$

Then we simplify by cancelling like terms on the left and applying the induction hypothesis on the right:

$$F_{ip^*p^m+1} + F_{p^*p^m-1}(-1)^{i-1}(i) \equiv_p (-1)^i + (-1)^i(i) \equiv_p (-1)^{(i+1)-1}(i+1).$$

□

Lemma 4.4. *For $i > 0$,*

$$\binom{ip^*p^m}{p^*p^m}_F \equiv_p i.$$

Proof. By definition of the fibonomial coefficient,

$$\binom{ip^*p^m}{p^*p^m}_F = \frac{F_{ip^*p^m}F_{ip^*p^m-1}\dots F_{(i-1)p^*p^m}F_{(i-1)p^*p^m-1}\dots F_1}{(F_{(i-1)p^*p^m}F_{(i-1)p^*p^m-1}\dots F_1)F_{p^*p^m}F_{p^*p^m-1}\dots F_1}$$

Cancelling like terms gives

$$\frac{F_{ip^*p^m}F_{ip^*p^m-1}\dots F_{ip^*p^m-(p^*p^m-1)}}{F_{p^*p^m}F_{p^*p^m-1}\dots F_1}$$

The terms in the above expression take three forms, which we represent separately for clarity. Note that all reduction modulo p happens term-wise, and thus the result is an integer.

(1) We first consider terms of the form $F_{ip^*p^m-a}$, where $p^* \nmid a$. Write

$$\prod_{a=1}^{p^*p^m-1} \frac{F_{ip^*p^m-a}}{F_{p^*p^m-a}}.$$

We apply Lemma 4.2 to the top $p^*p^m - 1$ many times so that we can cancel the top and bottom, resulting in $(-1)^{(i-1)(p^*p^m-1)} \equiv_p (-1)^{i-1}$, because $p^*p^m - 1$ is odd.

(2) Next we consider terms of the form $F_{(ip^m-a)p^*}$:

$$\prod_{a=1}^{p^m-1} \frac{F_{(ip^m-a)p^*}}{F_{(p^m-a)p^*}}.$$

By Lemma 4.1 and Lemma 4.3,

$$\left(\prod_{a=1}^{p^m-1} \frac{F_{(ip^m-a)p^*}}{F_{(p^m-a)p^*}}\right)\left(\frac{\frac{1}{F_{p^*}}}{\frac{1}{F_{p^*}}}\right)^{p^m-1} \equiv_p \prod_{a=1}^{p^m-1} \frac{(-1)^{ip^m-a-1}(ip^m-a)}{(-1)^{p^m-a-1}(p^m-a)} \equiv_p \prod_{a=1}^{p^m-1} \frac{(-1)^{(i-1)p^m}(-a)}{(-a)}.$$

Note that in the modular group we use division notation to represent multiplication by an inverse.

Since p^m is odd and p^m-1 is even,

$$\prod_{a=1}^{p^m-1} \frac{(-1)^{(i-1)p^m}(-a)}{(-a)} \equiv_p (-1)^{(i-1)(p^m-1)} \equiv_p 1.$$

(3) The only remaining term is the quotient $\frac{F_{ip^*p^m}}{F_{p^*p^m}} \equiv_p i(-1)^{i-1}$, by Lemma 4.3.

From the three cases above,

$$\binom{ip^*p^m}{p^*p^m}_F \equiv_p i(-1)^{2(i-1)} \equiv_p i$$

□

Lemma 4.5. *For* $0 \le i, j < p$,

$$\binom{ip^*p^m}{jp^*p^m}_F \equiv_p \binom{i}{j}.$$

Proof. We prove this using induction. For a base case, let $i = 0$. Then if $j = 0$,

$$\binom{0}{0}_F \equiv_p 1 \equiv_p \binom{0}{0}.$$

If $j > 0$, then

$$\binom{0}{jp^*p^m}_F \equiv_p 0 \equiv_p \binom{0}{j}.$$

Now assume $\binom{ip^*p^m}{jp^*p^m}_F \equiv_p \binom{i}{j}$ Then we apply Lemmas 2.5 and 2.6.

We let $a = (i+1)p^*p^m$, $b = (i+1-j)p^*p^m$, and $c = p^*p^m$, thus yielding the following:

$$\binom{(i+1)p^*p^m}{jp^*p^m}_F \binom{(i+1-j)p^*p^m}{p^*p^m}_F = \binom{(i+1)p^*p^m}{p^*p^m}_F \binom{ip^*p^m}{jp^*p^m}_F$$

Applying the induction hypothesis and Lemma 4.4 gives

$$\binom{(i+1)p^*p^m}{jp^*p^m}_F (i+1-j) \equiv_p (i+1)\binom{i}{j}.$$

We then multiply both sides by $(i+1-j)^{-1}$ to obtain

$$\binom{(i+1)p^*p^m}{jp^*p^m}_F \equiv_p \frac{(i+1)}{(i+1-j)}\binom{i}{j}.$$

Equivalently,

$$\binom{(i+1)p^*p^m}{jp^*p^m}_F \equiv_p \binom{i+1}{j},$$

as desired.

□

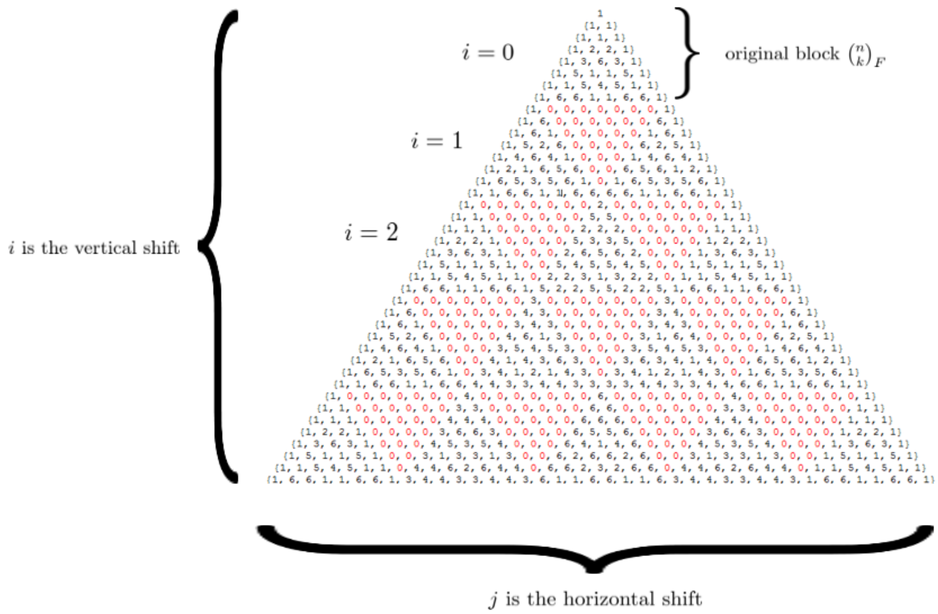

FIGURE 1. Exact congruence classes modulo p.

We now proceed with our main theorem for the exact congruence classes of the fibonomial triangle modulo p. For an visual representation of the relation, see Figure 1.

Theorem 4.6. *For $0 \le n, k < p^* p^m$, $0 \le i, j < p, 0 \le m$,*

$$\binom{n + ip^*p^m}{k + jp^*p^m}_F \equiv_p (-1)^{ik-nj} \binom{i}{j} \binom{n}{k}_F .$$

Proof. We proceed by induction.

First let $n = k = 0$. Then the statement follows directly from Lemma 4.5.

When $n = 0$, $k > 0$, by Theorem 3.3, since $p | \binom{0}{k}_F$,

$$\binom{ip^*p^m}{k + jp^*p^m}_F \equiv_p 0 \equiv_p (-1)^{ik-0j} \binom{i}{j} \binom{0}{k}.$$

Let $k, n > 0$. We assume

$$\binom{n - 1 + ip^*p^m}{k + jp^*p^m}_F \equiv_p (-1)^{ik-(n-1)j} \binom{i}{j} \binom{n-1}{k}_F$$

for all k.

Using the recurrence relation for fibonomial coefficients,

$$
\begin{aligned}
\binom{n+ip^*p^m}{k+jp^*p^m}_F &\equiv_p F_{n+(i-j)p^*p^m-k+1}\binom{n-1+ip^*p^m}{k-1+jp^*p^m}_F + F_{k-1+jp^*p^m}\binom{n-1+ip^*p^m}{k+jp^*p^m}_F \\
&\equiv_p (-1)^{i-j}F_{n-k+1}(-1)^{i(k-1)-(n-1)j}\binom{i}{j}\binom{n-1}{k-1}_F + (-1)^j F_{k-1}(-1)^{i(k)-(n-1)j}\binom{i}{j}\binom{n-1}{k}_F \\
&\equiv_p (-1)^{ik-nj}\binom{i}{j}\left[F_{n-k+1}\binom{n-1}{k-1}_F + F_{k-1}\binom{n-1}{k}_F\right] \\
&\equiv_p (-1)^{ik-nj}\binom{i}{j}\binom{n}{k}_F.
\end{aligned}
$$

This completes the proof.

□

5. Further Directions

In Theorem 4.6, note that

$$ik - nj = \det\begin{pmatrix} i & n \\ j & k \end{pmatrix}.$$

This may be a coincidence but, alternatively, it might indicate the existence of a more general relation for different types of primes.

Theorem 4.6 can be generalized to other primes by proving variants of the prerequisite lemmas. For example, in the case $p = 5$, $5^* = 5$, and $F_{n+5} \equiv_5 3F_n$ [7].

However, for some primes, problems arise. In the case $p = 11$, $p^* = 10$. In this case, one would need a base other than base $\mathcal{F}_{p^*}$, because the divisibility theorem cannot be proven in base $\mathcal{F}_{p^*}$ [7]. The form of such a base remains to be investigated.

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